Reduction by stages for affine W-algebras

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Vertex algebras

Vertex algebras are noncommutative and nonassociative algebras providing a rigorous mathematical framework for conformal field theories of dimension 2. They play a great role in various areas of mathematics: representation theory of infinite-dimensional Lie algebras, the Monstrous Moonshine Conjecture, the Langlands program, etc. Their axiomatic definition was given by Richard Borcherds in 1986.

Definition

Roughly speaking, a vertex algebra V is a vector space over the field of complex numbers \mathbf{C} , equipped with

- a multiplication $V \otimes_{\mathbf{C}} V \to V$, $a \otimes b \mapsto :ab$: (the normally ordered product),
- a unit vector $\mathbf{1}$ in V (the vacuum vector),
- a derivation operator $\partial: \mathcal{V} \to \mathcal{V}$ (the translation operator),
- $a \lambda$ -bracket $[a_{\lambda}b] = \sum_{n\geq 0} a_{(n)}b \frac{1}{n!}\lambda^n$ in $\mathcal{V}[\lambda]$ for any a, b in \mathcal{V} .

These data satisfy some axioms. For example, the λ -bracket controls the noncommutativity and nonassociativity of the normally ordered product.

Examples of vertex algebras

Virasoro vertex algebra

The Virasoro vertex algebra Vir^c with central charge given by the complex number c is spanned (as a differential algebra) by a vector L satisfying the relation

$$[L_{\lambda}L] = \partial L + 2L\,\lambda + \frac{c}{2}\mathbf{1}\,\frac{\lambda^3}{6}.$$

Kac-Moody vertex algebra

Let $\mathfrak g$ be a simple finite-dimensional Lie algebra over $\mathbf C$ and κ be an invariant bilinear form on $\mathfrak g$. The Kac-Moody vertex algebra $\mathcal V^\kappa(\mathfrak g)$ associated with $\mathfrak g$ and κ is spanned by elements x,y in $\mathfrak g$ with the relations

$$[x_{\lambda}y] = [x, y] + \kappa(x, y)\mathbf{1}\lambda.$$

Affine W-algebras

Let f be a nilpotent element in \mathfrak{g} . The affine W-algebra $\mathcal{W}^{\kappa}(\mathfrak{g}, f)$ associated \mathfrak{g} , f and κ is a vertex algebra constructed by applying a quantum Hamiltonian reduction functor H_f to the Kac–Moody vertex algebra $\mathcal{V}^{\kappa}(\mathfrak{g})$ [Feigin–Frenkel 1990 and Kac–Roan–Wakimoto 2003]:

$$W^{\kappa}(\mathfrak{g},f) := H_f(V^{\kappa}(\mathfrak{g})).$$

It is finitely generated as a differential algebra, but the relations between generators are unknown in general and difficult to compute.

The affine W-algebra $W^{\kappa}(\mathfrak{g}, f)$ only depends on the nilpotent orbit \mathbf{O} that contains f.

Example

If $\mathfrak{g} = \mathfrak{sl}_2$, if $\kappa(\bullet, \bullet) = k \operatorname{trace}(\bullet \bullet)$ for a complex number $k \neq -2$, and if f is a nonzero nilpotent element, then $\mathcal{W}^{\kappa}(\mathfrak{sl}_2, f)$ is a Virasoro vertex algebra with central charge given by $c_k = 1 - 6\frac{(k+1)^2}{k+2}$.

Reduction by stages

Let f_1 and f_2 be two nilpotent elements in $\mathfrak g$ such that $f_0:=f_2-f_1$ is nilpotent. We say that reduction by stages holds if there exists a quantum Hamiltonian reduction functor H_{f_0} that makes the following diagram commute:

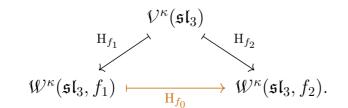
$$\mathcal{W}^{\kappa}(\mathfrak{g})$$
 $\mathcal{W}^{\kappa}(\mathfrak{g}, f_1) \longmapsto \mathcal{W}^{\kappa}(\mathfrak{g}, f_2).$

Motivation

Reduction by stages can be applied to the study of representations of the affine W-algebras and to provide isomorphisms between their simple quotients.

Example

Take $\mathfrak{g} = \mathfrak{sl}_3$, $f_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $f_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. The following reduction by stages holds:



Approach: using geometry

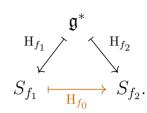
Associated variety

Arakawa introduced a (contravariant) functor $V \mapsto X_V$ from the category of vertex algebras to the category of affine Poisson varieties. The variety X_V is called the associated variety of V.

The associated variety of $V^{\kappa}(\mathfrak{g})$ is the dual space \mathfrak{g}^* with the Kirillov–Kostant Poisson structure. The associated variety of $W^{\kappa}(\mathfrak{g}, f)$ is the Slodowy slice S_f associated to f.

Strategy [GJ25]

Step 1. Prove the geometric reduction by stages:



- Step 2. Build a natural map $H_{f_0}(W^{\kappa}(\mathfrak{g}, f_1)) \to W^{\kappa}(\mathfrak{g}, f_2)$ by generalising ideas from [Madsen-Ragoucy 1997].
- Step 3. Prove it is an isomorphism by using the canonical Li filtrations on the W-algebras and the geometric reduction by stages.

Other approach: screening operators

Various reductions by stages were also proved using the screening operator description of affine W-algebras developed by Genra. See:

- [Fehily 2023 and 2024],
- [Fasquel-Nakatsuka 2023],
- [Fasquel-Fehily-Fursman-Nakatsuka 2024],
- [Fasquel-Kovalchuk-Nakatsuka 2024].

Main results

Theorem 1 [GJ24]

Reduction by stages holds for Slodowy slices in the cases described in Table 1.

Theorem 2 [GJ25]

Reduction by stages holds for affine W-algebras in the cases described in Table 1.

In type A, the nilpotent orbit of \mathfrak{sl}_n are classified by partitions $(a_1, \ldots, a_{r-1}, a_r)$ of n. A hook-type partition is a partition of the form $(a, 1^{n-a})$.

Table 1. Setting for reductions by stages

\mathfrak{g}		f_1	f_2	Reference
type	e A	hook-type	hook-type	[MR97, F24]
type	A_3	partition of 4: $(2,1^2)$	partition of 4: $(2,2)$	[FFFN24]
type A	Λ_{n-1}	partition of n : $(a_1, \ldots, a_{r-1}, a_r, 1^p)$	partition of n : $(a_1, \dots, a_{r-1}, a_r + 1, 1^{p-1})$	$\begin{array}{c} \text{new} \\ \text{for } n > 3 \end{array}$
type	В	subregular	regular	[FN23]
type	C_r	partition of r : $(2^2, 1^{2r-4})$	regular	new
type	G_2	Bala-Carter label $\widetilde{\mathrm{A}}_1$	regular	new

Application to the Kac-Roan-Wakimoto embedding

Let $(a_1, \ldots, a_s, a_{s+1}, \ldots, a_r)$ be a partition of n associated with the nilpotent element f_2 in \mathfrak{sl}_n . Set $p := a_{s+1} + \cdots + a_r$ and let f_1 be a nilpotent element corresponding to the partition $(a_1, \ldots, a_s, 1^p)$ of n. Let f_0 be a nilpotent element of \mathfrak{sl}_p corresponding to the partition (a_{s+1}, \ldots, a_r) of p.

Theorem [KRW03]

There is a level β such that there is a vertex algebra embedding $V^{\beta}(\mathfrak{sl}_p) \hookrightarrow W^{\kappa}(\mathfrak{sl}_n, f_1)$.

Corollary [GJ, in progress]

There is a vertex algebra embedding $W^{\beta}(\mathfrak{sl}_p, f_0) \hookrightarrow W^k(\mathfrak{sl}_n, f_2)$.

References

[GJ24] Naoki Genra and Thibault Juillard. Reduction by stages for finite W-algebras. Mathematische Zeitschrift, 308(1):15, 2024.

[GJ25] Naoki Genra and Thibault Juillard. Reduction by stages for affine W-algebras. 2025. Preprint.