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Reduction by stages for W-algebras and applications

Réduction par étapes pour les W-algèbres et applications

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THESE DE DOCTORAT

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Title: Reduction by stages for W-algebras and applications.

Keywords: Vertex algebra, W-algebra, Slodowy slice, BRST cohomology.

Abstract: *Affine W-algebra* form a family of vertex algebras defined as quantum Hamiltonian reductions of affine Kac–Moody algebras. They are noncommutative and nonassociative algebras of infinite type, in one-to-one correspondence with nilpotent orbits in simple Lie algebras. Any affine W-algebra has an associative and finitely generated analogue, a *finite W-algebra*. The algebraic properties of affine or finite W-algebra are related to the geometric properties of some affine Poisson variety, a *Slodowy slice*.

Given a pair of nilpotent orbits, one can associate of a pair of (finite or affine) W-algebras. In this thesis, under some compatibility conditions on these orbits, we prove that one of these two W-algebras can be reconstructed as the quantum Hamiltonian reduction of the other one. This property is called *reduction by stages*. To prove reduction by stages for W-algebras, we first prove reduction by stages for the pair of Slodowy slices associated with the chosen pair of nilpotent orbits.

Reduction by stages for finite W-algebras is proved by introducing filtrations on both W-algebras such that the associated graded algebras coincide with the Poisson algebras of polynomial functions on the Slodowy slices. As an application to to reduction by stages, we establish an analogue of the Skryabin equivalence of categories for the modules over these W-algebras.

For affine W-algebras, the quantum Hamiltonian reduction is a achieved by the mean of *BRST cohomology*, implying new technical difficulties. We need to prove that each W-algebra can be defined by using several equivalent BRST cohomology constructions. Then, choosing the right BRST construction allows us to connect the two affine W-algebras in a natural way and deduce reduction by stages.

We provide several examples of compatible pairs of nilpotent orbits for classical and exceptional simple Lie algebras. In type A, a fundamental example is when the chosen nilpotent orbits correspond to *hook-type partitions*. As an application, we give a new interpretation of the Kraft–Procesi isomorphisms between nilpotent Slodowy slices by using reduction by stages. We explain our future project of applying reduction by stages to prove isomorphisms of simple quotients of affine W-algebras that are analogous to these Kraft–Procesi isomorphisms.

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ÉCOLE DOCTORALE
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Titre : Réduction par étapes pour les W-algèbres et applications.
Mots clés : Algèbre vertex, W-algèbre, tranche de Slodowy, cohomologie BRST.

Résumé: Les *W-algèbres affines* forment une famille d'algèbres vertex définies comme réductions hamiltoniennes quantiques d'algèbres de Kac–Moody affines. Ce sont des algèbres non-commutatives et non-associatives, indexées par les orbites nilpotentes des algèbres de Lie simples. Toute W-algèbre affine a un analogue associatif et finiment engendré, une *W-algèbre finie*. Les propriétés algébriques d'une W-algèbre affine ou finie sont liées aux propriétés géométriques d'une variété de Poisson affine, une *tranche de Slodowy*.

Étant donnée une paire d'orbites nilpotentes, on peut considérer une paire de W-algèbres (finies ou affines). Dans cette thèse, sous certaines conditions de compatibilité entre ces orbites, nous démontrons que l'une de ces deux W-algèbres peut être reconstruite comme réduction hamiltonienne quantique de l'autre. Ce procédé est dénommé *réduction par étapes*. Pour établir la réduction par étapes pour les W-algèbres, nous démontrons d'abord que la réduction par étapes a lieu pour la paire de tranches de Slodowy associée à la paire d'orbites nilpotentes que l'on a choisies.

La réduction par étapes pour les W-algèbres finies est démontrée grâce à l'introduction de filtrations sur les deux W-algèbres telles que les algèbres graduées associées coïncident avec les algèbres de Poisson constituées des fonctions polynomiales sur les tranches de Slodowy. Comme application de la réduction par étapes, nous établissons un analogue de l'équivalence de catégorie de Skryabin pour les modules sur ces W-algèbres.

Pour les W-algèbres affines, la réduction hamiltonienne quantique est réalisée au moyen de la *cohomologie BRST*, entraînant de nouvelles difficultés techniques. Nous devons établir que chaque W-algèbre peut être définie en utilisant des constructions cohomologiques différentes mais équivalentes. Choisir les bonnes constructions nous permet de relier naturellement les deux W-algèbres et d'en déduire la réduction par étapes.

On produit plusieurs exemples de paires d'orbites nilpotentes compatibles pour des algèbres de Lie simples classiques et exceptionnelles. En type A, un exemples fondamental est quand les orbites nilpotentes correspondent à des *partitions en équerre*. En guise d'application, nous donnons une nouvelle interprétation des isomorphismes de Kraft–Procesi entre des tranches de Slodowy nilpotentes à l'aide de la réduction par étapes. Nous expliquons notre futur projet d'appliquer la réduction par étapes à la démonstration d'isomorphismes de quotients simples de W-algèbres affines qui sont analogues aux isomorphismes de Kraft–Procesi.

Contents

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Le chimiste Joe Hoover, dans le roman
La Nuit des Temps, par René Barjavel

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> "Sheldon, I've given you the simplest things to do, and you haven't done one of them right."
> "Maybe that's because I'm not being challenged. It's the same reason Einstein failed math."

Amy Farrah Fowler et Sheldon Cooper, dans le série
The Big Bang Theory, par Chuck Lorre et Bill Prady

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> « Deux ans ! dit Dantès, vous croyez que je pourrais apprendre toutes ces choses en deux ans ?
> — Dans leur application, non ; dans leurs principes, oui : apprendre n'est pas savoir ; il y a les sachants et les savants : c'est la mémoire qui fait les uns, c'est la philosophie qui fait les autres. »

Edmond Dantès et l'abbé Faria, dans le roman
Le Comte de Monte Cristo, par Alexandre Dumas

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> Lui on ne comprend ni c'qui fait, ni c'qui dit... C'est rare.
>
> Le paysan Guethenoc, dans la série *Kaamelott*, par Alexandre Astier

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> Questions of science, science and progress
> Do not speak as loud as my heart

Paroles de la chanson *The Scientist*,
par le groupe Coldplay

1 - Introduction

1.1. Introduction (en français)

Les W-algèbres affines forment une famille d'algèbres vertex graduées construites par réduction hamiltonienne quantique. Dans cette thèse, nous étudions une condition suffisante sur une paire de W-algèbres affines pour s'assurer que l'une d'entre elles peut être reconstruite en tant que réduction hamiltonienne quantique de l'autre. Cette propriété est appelée *réduction hamiltonienne par étapes*, par analogie avec la géométrie symplectique [MMO+07, Mor15a, Mor15b, GJ24]. Nos principales motivations sont : établir des connections entre des catégories de modules de diverses W-algèbres, construire des plongements entre W-algèbres ou des isomorphismes entre leurs quotients simples.

Chaque W-algèbre affine est en relation avec une variété de Poisson affine, la tranche de Slodowy associée, et avec une algèbre associative unitaire, appelée W-algèbre finie. La W-algèbre finie est aussi une quantification de la tranche de Slodowy. Notre stratégie est de démontrer la réduction par étapes pour une paire des tranches de Slodowy et de relever ce résultat pour les W-algèbres finies et affines correspondantes en utilisant des filtrations sur elles. Une grande partie de cette thèse provient de deux articles écrits par l'auteur en collaboration avec Naoki Genra [GJ24, GJ25].

1.1.1. W-algèbres affines

Les *algèbres vertex* sont des structures algébriques qui ont été axiomatisées par Borcherds [Bor86] pour démontrer la conjecture du Moonshine [FLM89, Bor92]. Ce sont des algèbres différentielles non-commutatives et non-associatives, mais dont l'absence de commutativité et d'associativité est contrôlé par les identités de Borcherds. Elles fournissent un cadre mathématique rigoureux pour définir la partie chirale d'une théorie conforme des champs [BPZ84]. Les algèbres de Virasoro et les algèbres de Kac–Moody affines sont des exemples d'algèbres vertex.

Les *W-algèbres affines* forment une famille importante d'algèbres vertex obtenues par réduction hamiltonienne quantique d'algèbres de Kac–Moody affines. Elles ont été introduites en physique théorique par Belavin, Polyakov et Zamolodchikov pour généraliser les symétries de Virasoro en théorie conforme des champs de dimension 2 [BPZ84, Zam85]. Les W-algèbres affines généralisent à la fois les algèbres de Lie de Virasoro et les algèbres de Lie de Kac–Moody affines, mais à cause de la non-linéarité dans les relations entre leurs générateurs, elles ne sont pas engendrées par des algèbres de Lie en général [BS93]. Les W-algèbres affines jouent aussi un rôle dans l'étude des théories conformes des champs en dimension 4 grâce à la *dualité* 4d/2d [BLL+15, Ara19, Ara18, SXY24], dans la *correspondance AGT* [SV13] ou le *programme de Langlands géométrique quantique* [Fre07, AF19].

La W-algèbre affine $\mathscr{W}^k(\mathfrak{g}, f)$ est une algèbre vertex définie à partir de la donnée d'une algèbre de Lie complexe simple de dimension finie $\mathfrak{g}$, d'un élément nilpotent f dans $\mathfrak{g}$ et d'un nombre complexe k. Si $f = 0$, alors la W-algèbre correspondante est l'*algèbre vertex affine universelle* $\mathscr{V}^k(\mathfrak{g})$ associée à $\mathfrak{g}$ et k, qui est l'analogue, en tant qu'algèbre vertex, de l'algèbre de Kac–Moody affine $\widehat{\mathfrak{g}}^k$. L'algèbre vertex affine universelle a été introduite par I. Frenkel et Zhu [FZ92]. En général, $\mathscr{W}^k(\mathfrak{g}, f)$ est construite en appliquant à $\mathscr{V}^k(\mathfrak{g})$ un foncteur de cohomologie BRST, noté H^0_f:

$$\mathscr{W}^k(\mathfrak{g}, f) \coloneqq \mathrm{H}^0_f(\mathscr{V}^k(\mathfrak{g})).$$

La *cohomologie BRST (Becchi–Rouet–Stora–Tyutin)* a été développée par Kostant et Sternberg [KS87] pour calculer des réductions hamiltoniennes dans le cadre non-commutatif, en formalisant la méthode de quantification BRST issues de la physique théorique. Rappelons qu'étant données l'action d'un groupe de Lie sur une variété différentielle symplectique et une application moment, on peut réaliser une réduction hamiltonienne, qui consiste à faire le quotient, par l'action du groupe, de la fibre de zéro par l'application moment. Sous certaines bonnes hypothèses, ce quotient a une structure naturelle de variété différentielle symplectique [MW74]. La cohomologie BRST, appliquée à l'anneau des fonctions de la variété que l'on veut réduire, produit l'anneau des fonctions de la variété réduite.

La construction BRST de $\mathscr{W}^k(\mathfrak{g}, f)$ est due à Feigin et E. Frenkel [FF90] pour f un élément nilpotent régulier (c'est-à-dire que son orbite adjointe est dense dans le cône nilpotent) et à Kac, Roan et Wakimoto en général [KRW03, KW04]. Ce sont les algèbres enveloppantes affines universelles de certaines algèbres de Lie (conformes) non-linéaires [DSK05] : cela signifie qu'elles sont librement engendrées avec des relations non-linéaires entre les générateurs. En général, ces générateurs et relations sont très difficiles à calculer. Cela a été fait quand f est un élément nilpotent minimal [KW04] ou pour des exemples en petits rangs [CL18, Fas22, Fas25]. Si $\mathfrak{g}$ est $\mathfrak{sl}_2$ et f n'est pas zéro, alors $\mathscr{W}^k(\mathfrak{sl}_2, f)$ est une algèbre vertex de Virasoro quand $k + 2 \neq 0$.

1.1.2. Réduction par étapes

Étant donnée l'action d'un groupe de Lie sur une variété symplectique, quand le groupe contient un sous-groupe fermé normal, il est possible de réaliser la réduction Hamiltonienne en deux étapes. On peut d'abord faire la réduction Hamiltonienne par le sous-groupe normal (première étape). Puis, on peut considérer l'action induite du quotient des deux groupes sur la variété symplectique réduite et refaire une réduction hamiltonienne (seconde étapes). Ce procédé est appelé *réduction hamiltonienne par étapes* [MMO+07]. Sous de bonnes hypothèses, la réduction hamiltonienne par le groupe ambiant et la réduction par étapes produisent la même variété symplectique, à un isomorphisme naturel près.

Dans le contexte des W-algèbres affines, cette construction a un analogue naturel avec la cohomologie BRST. Soient f_1, f_2 deux éléments nilpotents dans $\mathfrak{g}$ et notons $\mathbf{O}_1, \mathbf{O}_2$ leur orbites adjointes. Supposons l'inclusion $\overline{\mathbf{O}_1} \subseteq \overline{\mathbf{O}_2}$ de leurs adhérences de Zariski, et supposons que l'élément $f_0 \coloneqq f_2 - f_1$ est nilpotent. On dit que la *réduction par étapes* a lieu pour la paire de W-algèbres associées s'il existe un foncteur de cohomologie BRST $\mathrm{H}^0_{f_0}$ qui peut être appliqué à la première W-algèbre affine pour obtenir la seconde, à un isomorphisme naturel d'algèbres vertex près:

$$\mathrm{H}^0_{f_0}(\mathscr{W}^k(\mathfrak{g}, f_1)) \cong \mathscr{W}^k(\mathfrak{g}, f_2).$$

Autrement dit, on a le triangle commutatif:

$$\begin{tikzcd} & \mathscr{V}^k(\mathfrak{g}) \arrow[dl, mapsto, "\mathrm{H}^0_{f_1}"'] \arrow[dr, mapsto, "\mathrm{H}^0_{f_2}"] & \\ \mathscr{W}^k(\mathfrak{g}, f_1) \arrow[rr, mapsto, "\mathrm{H}^0_{f_0}"'] & & \mathscr{W}^k(\mathfrak{g}, f_2). \end{tikzcd}$$

La réduction par étapes a été étudiée par Madsen et Ragoucy dans l'article de physique théorique [MR97], quand l'algèbre de Lie simple $\mathfrak{g}$ est $\mathfrak{sl}_n$ et les éléments nilpotents f_1, f_2 correspondent à des *partitions en équerre*, autrement dit des partitions de la forme $(a, 1^{n-a})$ où $1 \leqslant a \leqslant n$. Leur approche consiste à démontrer l'isomorphisme

$$\mathrm{H}^0_{f_0}(\mathrm{H}^0_{f_1}(\mathscr{V}^k(\mathfrak{g}))) \cong \mathrm{H}^0_{f_2}(\mathscr{V}^k(\mathfrak{g}))$$

en remarquant que le membre de droite est la cohomologie du complexe total d'un complexe double, et que le membre de gauche est la seconde page de la *suite spectrale* associée. Nous allons généraliser cette démarche.

Récemment, des réductions par étapes de W-algèbres affines (aussi appelées *réduction partielles*) ont été étudiées avec intensité en utilisant des réalisations en champs libres des W-algèbres affines développées dans [Gen20]. En général, cette approche permet d'étudier la réduction par étapes avec sa *réduction inverse* associée. La réduction inverse consiste à « inverser » le foncteur de cohomologie BRST H^0_f, au sens où on construit un plongement d'algèbres vertex

$$\mathscr{V}^k(\mathfrak{g}) \longhookrightarrow \mathscr{W}^k(\mathfrak{g}, f) \otimes_{\mathbf{C}} \mathscr{D}_{\mathrm{ch}},$$

où on note $\mathscr{D}_{\mathrm{ch}}$ l'algèbre vertex des opérateurs différentiels chiraux sur une variété affine de la forme $\mathbf{A}^{n_1} \times \mathbf{G}_{\mathrm{m}}^{n_2}$. Ces plongements sont des outils utiles pour la théorie des représentations des W-algèbres affines [Ada19, AKR21, ACG24].

Cette idée se généralise à la réduction par étapes. Dans ce contexte, une réduction inverse est un plongement de la forme

$$\mathcal{W}^k(\mathfrak{g}, f_1) \longrightarrow \mathcal{W}^k(\mathfrak{g}, f_2) \otimes_{\mathbf{C}} \mathcal{D}_{\mathrm{ch}}.$$

Les réductions par étapes (inverses) ont été étudiées dans [Feh23, Feh24] en type A pour des éléments nilpotents en équerre; dans [FN23] en type B pour des éléments nilpotents sous-régulier et régulier; dans [CFLN24] en type A et petits rangs; dans [FFFN24] pour $\mathfrak{g} = \mathfrak{sl}_4$ et toute paire ordonnée d'orbites nilpotentes; et dans [FKN24] en type classique pour H_{f_0} une réduction de type Virasoro. D'après [But23, Section 5], les réductions par étapes inverses en type A pourraient découler de la description des W-algèbres affines commes des algèbres vertex associées à un diviseur dans une 3-variété de Calabi–Yau [But23, Theorem 5.9].

1.1.3. Techniques de quantification

Étant donnée une algèbre non-commutative avec une filtration, il arrive souvent que l'algèbre graduée associée soit commutative et équipée d'un crochet de Poisson hérité du crochet non-trivial de l'algèbre filtrée. Alors l'algèbre graduée est l'algèbre des fonctions sur une variété de Poisson affine (ou plus généralement un schéma) avec une action du groupe multiplicatif $\mathbf{G}_{\mathrm{m}}$ [BPW16]. On dit alors que l'algèbre non-commutative est une *quantification* du schéma de Poisson affine. Une grande idée de la théorie géométrique des représentations est que l'algèbre filtrée et sa théorie des représentations peuvent être mieux comprises en étudiant les propriétés géométriques de ce schéma de Poisson.

Un exemple classique de quantification est l'anneau des opérateurs différentiels sur une variété holomorphe, qui quantifie l'espace cotangent de cette variété. Plus généralement, tout D-module est connecté à la géométrie d'une sous-variété de l'espace cotangent, appelée variété caractéristique de ce D-module [Bor87, Kas03]. Si $\mathcal{D}(\mathbf{A}^n)$ est l'algèbre des opérateurs différentiels algébriques sur l'espace affine $\mathbf{A}^n$, elle est filtrée par l'ordre des opérateurs différentiels et l'algèbre graduée $\operatorname{gr}\mathcal{D}(\mathbf{A}^n)$ est une algèbre commutative de Poisson isomorphe à l'algèbre des fonctions sur l'espace cotangent $\mathrm{T}^*\mathbf{A}^n$.

Un autre exemple, plus proche des motivations de cette thèse, est l'algèbre universelle enveloppante $\mathcal{U}(\mathfrak{g})$ d'une algèbre de Lie complexe de dimension finie $\mathfrak{g}$. Par le théorème de Poincaré–Birkhoff–Witt, cette algèbre est filtrée et quantifie l'espace dual $\mathfrak{g}^*$ équipé de la structure de Poisson de Kirillov–Kostant. Si $\mathfrak{g}$ est simple, d'après les travaux de Joseph, les représentations irréductibles de plus haut poids de $\mathfrak{g}$ sont des quantifications des adhérences d'orbites nilpotentes dans $\mathfrak{g}^*$ [Jos85].

Il a été établi par Li que toute algèbre vertex est équipée d'une filtration naturelle telle que l'algèbre graduée associée est une algèbre différentielle commutative avec une structure de Poisson vertex [Li05]. Arakawa a montré que pour tout schéma de Poisson affine, l'algèbre des fonctions de l'espace des arcs associé est aussi équippée d'une structure de Poisson vertex [Ara12]. Suivant les idées de Zhu, il a défini le *schéma associé* $X_{\mathscr{V}}$ d'une algèbre vertex (finiment engendrée) $\mathscr{V}$. En général, $\mathrm{gr}^{\mathrm{Li}}\,\mathscr{V}$ est un quotient de $\mathbf{C}[\mathrm{J}_\infty X_{\mathscr{V}}]$, l'algèbre des fonctions de l'espace des arcs du schéma associé.

Par exemple, pour l'algèbre vertex affine universelle $\mathscr{V}^k(\mathfrak{g})$, l'algèbre graduée associée $\mathrm{gr}^{\mathrm{Li}}\,\mathscr{V}^k(\mathfrak{g})$ est isomorphe à l'algèbre des fonctions sur l'espace des arcs $\mathrm{J}_\infty\mathfrak{g}^*$ de $\mathfrak{g}^*$ en tant qu'algèbres de Poisson vertex. Notons $\mathscr{L}_k(\mathfrak{g})$ l'unique quotient simple gradué de $\mathscr{V}^k(\mathfrak{g})$, appelé *algèbre vertex affine simple*. Si le nombre complexe k est *admissible* (voir la section 6.3.3 pour des exemples en type A), alors le schéma réduit associé à $X_{\mathscr{L}_k(\mathfrak{g})}$ est l'adhérence d'une orbite nilpotente qui dépend du niveau k, notée $\mathbf{O}_k$ [Ara15].

1.1.4. Tranches de Slodowy et W-algèbres finies

Il a été établi que la filtration de Li relie les W-algèbres affines aux espaces des arcs de certaines variétés affines, les *tranches de Slodowy*, qui ont été introduites par Slodowy pour étudier les singularités des adhérences des orbites nilpotentes dans une algèbre de Lie simple $\mathfrak{g}$ [Slo06]. À tout élément nilpotent f, on peut associer une tranche de Slodowy S_f qui est un sous-espace affine de $\mathfrak{g}^*$ transverse à n'importe quelle orbite co-adjointe qu'elle intersecte. Cette transversalité fait que S_f a une structure de Poisson induite par les structures symplectiques sur les orbites.

Gan et Ginzburg ont démontré que la structure de Poisson sur S_f peut aussi être construite par réduction hamiltonienne de $\mathfrak{g}^*$ [GG02]. Le groupe agissant N est un groupe unipotent dont l'algèbre de Lie $\mathfrak{n}$ est une sous-algèbre de la partie positive d'une graduation $\mathfrak{g} = \bigoplus_{\delta\in\mathbf{Z}}\mathfrak{g}_\delta$ qui est *bonne* pour f au sens de [EK05, BG07]. Leur construction est une réinterprétation et la suite des résultats de Premet, qui a construit la structure de Poisson sur les tranches de Slodowy comme induite par le crochet de la W-algèbre finie correspondante [Pre02]. Ces résultats généralisent ceux obtenus par Kostant dans le cas d'un élément nilpotent régulier [Kos78] et par Lynch pour les éléments nilpotents pairs [Lyn79].

La *W-algèbre finie* $\mathcal{U}(\mathfrak{g}, f)$ correspondante a d'abord été étudiée par Kostant pour un élément nilpotent régulier f [Kos78], puis Premet a fourni une construction en général par réduction hamiltonienne quantique [Pre02]. Premet a aussi défini une filtration telle que $\mathrm{gr}\,\mathcal{U}(\mathfrak{g}, f)$ est une algèbre commutative isomorphe à $\mathbf{C}[S_f]$. Gan et Ginzburg ont prouvé que cet isomorphisme est compatible avec les structures de Poisson des deux côtés, donc la W-algèbre finie quantifie la tranche de Slodowy. Cela a été généralisé au cadre des bonnes graduations par Brundan et Goodwin dans [BG07], et par Sadaka aux graduations admissibles [Sad16].

La tranche de Slodowy est liée à la W-algèbre affine car $\mathrm{gr}^{\mathrm{Li}}\,\mathscr{W}^k(\mathfrak{g}, f)$ et $\mathbf{C}[\mathrm{J}_\infty S_f]$ sont isomorphes en tant qu'algèbres de Poisson vertex [DSK06, Ara15]. Cette relation a été beaucoup utilisée pour étudier la structure et la

Figure 1.1

$$
\begin{tikzcd}
\mathscr{V}^k(\mathfrak{g}) \arrow[r, mapsto, "\mathrm{gr}^{\mathrm{Li}}"] \arrow[d, mapsto, "\mathrm{Zhu}"'] & \mathbf{C}[\mathrm{J}_\infty \mathfrak{g}^*] \arrow[d, mapsto, "\mathrm{Zhu}"] \arrow[l, dashed, bend right] \\
\mathcal{U}(\mathfrak{g}) \arrow[r, mapsto, "\mathrm{gr}"'] & \mathbf{C}[\mathfrak{g}^*] \arrow[l, dashed, bend left]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathscr{W}^k(\mathfrak{g}, f) \arrow[r, mapsto, "\mathrm{gr}^{\mathrm{Li}}"] \arrow[d, mapsto, "\mathrm{Zhu}"'] & \mathbf{C}[\mathrm{J}_\infty S_f] \arrow[d, mapsto, "\mathrm{Zhu}"] \arrow[l, dashed, bend right] \\
\mathcal{U}(\mathfrak{g}, f) \arrow[r, mapsto, "\mathrm{gr}"'] & \mathbf{C}[S_f] \arrow[l, dashed, bend left]
\end{tikzcd}
$$

théorie des représentations de la W-algèbre affine, voir [Ara15, AM18, AvEM24, AM25]. Les W-algèbres affine et finie sont reliées par la construction de l'algèbre de Zhu [Ara07, DSK06]. Nous donnons un aperçu de toutes ces algèbres et leurs relations dans la figure 1.1, où les flèches en pointillés représentent des quantifications.

Le problème de réduction par étapes pour les tranches de Slodowy et les W-algèbres finies a été d'abord étudié par Morgan dans sa thèse de doctorat [Mor15a, Mor15b]. Soient f_1, f_2 deux éléments nilpotents dans $\mathfrak{g}$ et notons $\mathbf{O}_1, \mathbf{O}_2$ leurs orbites adjointes. Supposons l'inclusion $\overline{\mathbf{O}_1} \subseteq \overline{\mathbf{O}_2}$ et supposons que $f_0 \coloneqq f_2 - f_1$ est nilpotent. Il a été conjecturé par Morgan que le triangle suivant est commutatif [Mor15b, Objective 3.6] :

$$
\begin{tikzcd}
& \mathcal{U}(\mathfrak{g}) \arrow[dl, mapsto, "\mathrm{H}^0_{f_1}"'] \arrow[dr, mapsto, "\mathrm{H}^0_{f_2}"] & \\
\mathcal{U}(\mathfrak{g}, f_1) \arrow[rr, mapsto, "\mathrm{H}^0_{f_0}"'] & & \mathcal{U}(\mathfrak{g}, f_2),
\end{tikzcd}
$$

avec une conjecture analogue pour les tranches de Slodowy.

En fait, Morgan a découvert qu'il faut imposer certaines conditions sur les éléments nilpotents f_1, f_2 pour établir la réduction par étapes. En type A, pour toute paire d'orbites nilpotentes adjacentes pour l'ordre des adhérences, il a construit un groupe N_0 avec une action hamiltonienne sur la tranche de Slodowy S_{f_1} qui satisfait ces conditions. Il a alors conjecturé que la réduction hamiltonienne résultante est isomorphe à la tranche de Slodowy S_{f_2}, ce qu'il a démontré dans des cas particuliers (orbites sous-régulière et régulière) et pour quelques exemples de bas rang : voir [Mor15b, Conjecture 3.13] et le développement écrit juste après. Mentionnons que la réduction par étapes apparaît aussi dans le contexte des tranches dans la grassmanienne affine [KPW22], dont on sait qu'elles sont liées aux tranches de Slodowy en type A par l'isomorphisme de Mirković–Vybornov [MV07].

1.1.5. Contributions de cette thèse

Soit $\mathfrak{g}$ une algèbre de Lie simple et $\mathfrak{h}$ une sous-algèbre de Cartan de $\mathfrak{g}$. Pour $i = 1, 2$, soit H_i dans $\mathfrak{h}$ tel que la graduation associée

$$\mathfrak{g} = \bigoplus_{\delta \in \mathbf{Z}} \mathfrak{g}_\delta^{(i)}, \quad \text{où} \quad \mathfrak{g}_\delta^{(i)} := \{x \in \mathfrak{g} \mid [H_i, x] = \delta x\},$$

soit une bonne graduation (Définition 2.2.1.3) pour l'élément nilpotent f_i.

Puisque $[H_1, H_2] = 0$, on obtient une bi-graduation

$$\mathfrak{g} = \bigoplus_{\delta_1, \delta_2 \in \mathbf{Z}} \mathfrak{g}_{\delta_1, \delta_2} \quad \text{où} \quad \mathfrak{g}_{\delta_1, \delta_2} := \mathfrak{g}_{\delta_1}^{(1)} \cap \mathfrak{g}_{\delta_2}^{(2)}.$$

Posons $f_0 := f_2 - f_1$. Nos conditions suffisantes pour la réduction par étapes sont définies par :

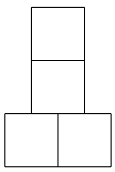
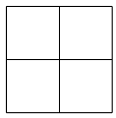
$$(\bigstar) \qquad \begin{cases} \mathfrak{g}_{\geqslant 2}^{(1)} \subseteq \mathfrak{g}_{\geqslant 1}^{(2)} \subseteq \mathfrak{g}_{\geqslant 0}^{(1)}, \quad \mathfrak{g}_1^{(1)} \subseteq \bigoplus_{\delta=0}^{2} \mathfrak{g}_{1,\delta}, \quad \mathfrak{g}_1^{(2)} \subseteq \bigoplus_{\delta=0}^{2} \mathfrak{g}_{\delta,1}, \\ f_0 \in \mathfrak{g}_{0,-2}. \end{cases}$$

Voir la table 1.1 pour des exemples où les conditions ($\bigstar$) sont vérifiées. On donne aussi l'exemple suivant comme fil rouge.

Exemple (Exemples 2.3.1.5 et 6.2.1.5). Prenons $\mathfrak{g} = \mathfrak{sl}_4$. On considère les éléments nilpotents

$$f_1 := \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ 1 & & & 0 \end{pmatrix} \quad \text{et} \quad f_2 := \begin{pmatrix} 0 & & & \\ & 0 & & \\ & 1 & 0 & \\ 1 & & & 0 \end{pmatrix}$$

respectivement dans les orbites associées aux diagrammes de Young suivants :

Alors les conditions ($\bigstar$) sont vérifiées pour f_1 et f_2.

Les conditions ($\bigstar$) sont introduites dans [GJ25]. Ces conditions impliquent l'inclusion $\overline{\mathbf{O}_1} \subseteq \overline{\mathbf{O}_2}$ des adhérences des orbites nilpotentes (Proposition 2.3.2.2). On peut construire un groupe N_0 (Proposition 2.3.1.1), une action de N_0 sur la tranche de Slodowy S_{f_1} et une application moment $\mu_0 : S_{f_1} \to \mathfrak{n}_0{}^*$, où $\mathfrak{n}_0$ désigne l'algèbre de Lie de N_0, tels que le théorème suivant soit vérifié.

Théorème 1 ([GJ24, Theorems 1 and 2], Théorème 2.3.3.3)**.** *Si les conditions ($\bigstar$) sont vérifiées, alors:*

1. *l'action de N_0 induit un isomorphisme $N_0 \times S_{f_2} \cong {\mu_0}^{-1}(0)$ qui est N_0-équivariant, où le membre de gauche est équipé de l'action par multiplication à gauche sur N_0,*

2. *on a un isomorphisme de Poisson*
$$S_{f_2} \cong {\mu_0}^{-1}(0)//N_0.$$

Il y a un plongement d'algèbres de Lie $\mathfrak{n}_0 \hookrightarrow \mathcal{U}(\mathfrak{g}, f_1)$ (Lemme 4.3.1.2). On peut donc considérer la réduction hamiltonienne quantique de $\mathcal{U}(\mathfrak{g}, f_1)$ par rapport à l'action de N_0. En introduisant des filtrations sur ces objets non-commutatifs, nous pouvons utiliser le théorème 1 pour établir le résultat suivant.

Théorème 2 ([GJ24, Theorem 3], Théorème 4.3.1.3)**.** *Si les conditions (★) sont vérifiées, alors la réduction hamiltonienne quantique de $\mathcal{U}(\mathfrak{g}, f_1)$ par rapport à l'action de N_0 est isomorphe à $\mathcal{U}(\mathfrak{g}, f_2)$ en tant qu'algèbres.*

Le théorème 2 découle du théorème 1 en introduisant des filtrations bien choisies sur $\mathcal{U}(\mathfrak{g}, f_1)$ et $\mathcal{U}(\mathfrak{g}, f_2)$. En guise d'application, nous obtenons un analogue de l'équivalence de Skryabin [Skr02]. En effet, l'élément nilpotent f_0 définit un caractère $\overline{\chi}_0$ de $\mathfrak{n}_0$, donc a une notion bien définie de catégorie des $\mathcal{U}(\mathfrak{g}, f_1)$-modules de Whittaker, notée Wh_0 : ce sont les $\mathcal{U}(\mathfrak{g}, f_1)$-modules à gauche V tels que pour tout x dans $\mathfrak{n}_0$, $x + \overline{\chi}_0(x)1$ agit localement de manière nilpotente. On a l'équivalence de catégories suivante.

Théorème 3 ([GJ24, Theorem 4], Proposition 4.4.3.3)**.** *On a l'équivalence de catégories*

$$\begin{tikzcd} \mathsf{Wh}_0 \arrow[r, bend left] \arrow[r, phantom, "\simeq"] & \mathcal{U}(\mathfrak{g}, f_2)\text{-}\mathsf{Mod}. \arrow[l, bend left] \end{tikzcd}$$

En fait, nous généralisons les arguments de [GG02] pour démontrer un analogue de l'équivalence de Skryabin dans un cadre très général, voir le théorème 4.4.2.2.

Remarque. L'étude des foncteurs de l'équivalence dans le théorème 3 a été poursuivie par Masut dans [Mas25]. Elle a démontré que l'équivalence est équivariante par rapport à des bi-actions d'une sous-catégorie monoïdale de $\mathcal{U}(\mathfrak{g})$-$\mathsf{Mod}$ sur les catégories Wh_0 et $\mathcal{U}(\mathfrak{g}, f_2)$-$\mathsf{Mod}$.

Le théorème 1 implique le même énoncé pour les espaces des arcs des variétés en jeu (Théorème 5.3.2.1). Cet énoncé a un analogue non-commutatif pour les W-algèbres affines qui quantifient ces espaces des arcs.

Théorème 4 ([GJ25, Theorem 1], Théorème 6.2.1.3)**.** *Si les conditions (★) sont vérifiées, alors il y a un complexe de co-chaînes BRST $C^\bullet_{f_0}(\mathcal{W}^k(\mathfrak{g}, f_1))$ dont la cohomologie est isomorphe à $\mathcal{W}^k(\mathfrak{g}, f_2)$ en tant qu'algèbres vertex :*
$$\mathrm{H}^\bullet\big(C^\bullet_{f_0}(\mathcal{W}^k(\mathfrak{g}, f_1))\big) \cong \delta_{\bullet=0}\, \mathcal{W}^k(\mathfrak{g}, f_2).$$

Exemples (Chapitre 3). Le théorème 4 est vrai dans les cas décrits dans la table 1.1 parce que les conditions (★) sont alors vérifiées. Par convention, les partitions qui indexent les orbites nilpotentes en types classiques sont représentées par des suites décroissantes. La colonne « Référence » donne la première occurrence dans la littérature de chaque réduction par étapes.

Table 1.1: Exemples de réductions par étapes

$\mathfrak{g}$	f_1	f_2	Référence
type A	équerre	équerre	[MR97, Feh24]
type A_3	partition de 4 : $(2, 1^2)$	partition de 4 : $(2, 2)$	[CFLN24]
type A_{n-1}	partition de n : $(a_1, \dots, a_{r-1}, a_r, 1^p)$	partition de n : $(a_1, \dots, a_{r-1}, a_r + 1, 1^{p-1})$	nouveau pour $n > 3$
type B	sous-régulier	régulier	[FN23]
type C_r	partition de r : $(2^2, 1^{2r-4})$	régulier	nouveau
type G_2	classe de Bala-Carter $\widetilde{A}_1$	régulier	nouveau

Pour établir le théorème 4, nous démontrons deux résultats importants qui ont leur intérêt en soi. Le premier est la définition d'un nouveau complexe BRST, noté $\widetilde{C}^{\bullet}_{f_2}(\mathcal{V}^k(\mathfrak{g}))$, pour reconstruire la W-algèbre $\mathcal{W}^k(\mathfrak{g}, f_2)$, avec un plongement

$$C^{\bullet}_{f_0}(\mathcal{W}^k(\mathfrak{g}, f_1)) \hookrightarrow \widetilde{C}^{\bullet}_{f_2}(\mathcal{V}^k(\mathfrak{g})).$$

Dés lors, cela induit un homomorphisme d'algèbres vertex

$$\Theta : \mathrm{H}^0\big(C^{\bullet}_{f_0}(\mathcal{W}^k(\mathfrak{g}, f_1))\big) \longrightarrow \mathrm{H}^0\big(\widetilde{C}^{\bullet}_{f_2}(\mathcal{V}^k(\mathfrak{g}))\big).$$

Théorème 5 ([GJ25, Theorem 2], Théorème 6.2.3.7)**.** *La cohomologie du complexe $\widetilde{C}^{\bullet}_{f_2}(\mathcal{V}^k(\mathfrak{g}))$ est isomorphe à la W-algèbre affine $\mathcal{W}^k(\mathfrak{g}, f_2)$:*

$$\mathrm{H}^{\bullet}\big(\widetilde{C}^{\bullet}_{f_2}(\mathcal{V}^k(\mathfrak{g}))\big) \cong \delta_{\bullet=0}\, \mathcal{W}^k(\mathfrak{g}, f_2).$$

Remarque. Le théorème 5 peut être comparé aux travaux de [Sad16] parce qu'il fournit des exemples de constructions de W-algèbres affines qui ne reposent pas sur des bonnes graduations, contrairement aux constructions habituelles [Pre02, GG02, BG07, KRW03].

Dés lors on obtient un homomorphisme d'algèbres vertex

$$\Theta : \mathrm{H}^0\Big(C^{\bullet}_{f_0}(\mathcal{W}^k(\mathfrak{g}, f_1))\Big) \longrightarrow \mathcal{W}^k(\mathfrak{g}, f_2).$$

Pour en déduire le théorème 5, c'est-à-dire pour prouver que l'homomorphisme d'algèbres vertex Θ est un isomorphisme, nous devons utiliser la filtration de Li et le théorème 1. L'application graduée associée $\operatorname{gr}\Theta$ est un isomorphisme en conséquence de la réduction par étapes pour les tranches de Slodowy associées S_{f_1}, S_{f_2} et leurs espaces des arcs. La filtration de Li n'est pas une filtration complète sur les complexes de co-chaînes BRST, on a donc un problème de convergence. Pour les surpasser, on utilise une généralisation de [AM25, Theorem 9.7].

Théorème 6 ([GJ25, Theorem 3], Théorème 5.3.5.1)**.** *Soit $(\mathscr{C}^\bullet, \mathscr{d})$ un complexe de co-chaînes BRST d'algèbres vertex. Supposons que quelques conditions techniques soient vérifiées, telles l'existence de certaines graduations et des bonnes propriétés géométriques pour avoir l'annulation de la cohomologie du complexe de co-chaînes gradué associé $(\operatorname{gr}^{\mathrm{Li}} \mathscr{C}^\bullet, \operatorname{gr}^{\mathrm{Li}} \mathscr{d})$.*

Alors, la cohomologie s'annule hors du degré 0 :

$$\mathrm{H}^n(\mathscr{C}^\bullet, \mathscr{d}) = 0 \quad \textit{pour} \quad n \neq 0.$$

En degré 0 on a un isomorphisme naturel

$$\operatorname{gr}^{\mathrm{F}} \mathrm{H}^0(\mathscr{C}^\bullet, \mathscr{d}) \xrightarrow{\sim} \mathrm{H}^0(\operatorname{gr}^{\mathrm{Li}} \mathscr{C}^\bullet, \operatorname{gr}^{\mathrm{Li}} \mathscr{d}),$$

où la filtration F sur la cohomologie $\mathrm{H}^0(\mathscr{C}^\bullet, \mathscr{d})$ est induite par la filtration de Li sur le complexe $(\mathscr{C}^\bullet, \mathscr{d})$.

Le théorème 6 est vérifié pour tous les complexes mentionnés jusqu'à présent, nous permettant de montrer le théorème 5, et donc le théorème 4.

Remarque. Une autre conséquence de ce théorème est que toutes les constructions possibles de $\mathscr{W}^k(\mathfrak{g}, f)$ par cohomologie BRST sont équivalentes : voir le théorème 6.1.2.6 ou la remarque avant le théorème 9.7 dans [AM25]. La démonstration s'inspire de [AKM15], où l'équivalence des définitions est démontrée pour les W-algèbres $\hbar$-adiques $\mathscr{W}^k(\mathfrak{g}, f)^\wedge_\hbar$.

1.1.6. Motivations et futurs travaux

Notre prochain objectif est d'étendre la réduction par étapes à tout module dans la catégorie de Kazhdan–Lusztig $\mathsf{KL}^k(\mathfrak{g})$ (Conjecture 6.3.1.2), c'est-à-dire construire un isomorphisme

$$\mathrm{H}^0_{f_0} \circ \mathrm{H}^0_{f_1}(M) \cong \mathrm{H}^0_{f_2}(M)$$

pour tout $\mathscr{V}^k(\mathfrak{g})$-module M dans $\mathsf{KL}^k(\mathfrak{g})$. Un tel résultat sera un moyen efficace de construire des homomorphismes entre W-algèbres.

Il a été établi dans [KRW03] que chaque W-algèbre affine $\mathscr{W}^k(\mathfrak{g}, f)$ contient une sous-algèbre isomorphe à une algèbre universelle affine, notée $\mathscr{V}^\ell(\mathfrak{a})$ (voir la proposition 6.1.4.1). Si la réduction par étapes a lieu dans une catégorie de modules suffisamment grosse, alors pour un élément nilpotent f_0 dans $\mathfrak{a}$ nous pouvons appliquer le foncteur $\mathrm{H}^0_{f_0}$ au plongement

$$\mathcal{V}^\ell(\mathfrak{a}) \hookrightarrow \mathcal{W}^k(\mathfrak{g}, f),$$

et en déduire un plongement de W-algèbres affines (voir la conjecture 6.3.3.1) :

$$\mathcal{W}^\ell(\mathfrak{a}, f_0) \hookrightarrow \mathcal{W}^k(\mathfrak{g}, f + f_0). \tag{1.1.6.1}$$

Le plongement (1.2.6.1) est important parce que pour des valeurs particulières de k (et de ℓ), il induit des isomorphismes entre les quotients simples de ces W-algèbres affines (Conjecture 6.3.3.3)

$$\mathcal{W}_\ell(\mathfrak{a}, f_0) \cong \mathcal{W}_k(\mathfrak{g}, f + f_0).$$

Le niveau k is alors dit d'*effondrement*. Le fait pour un niveau k d'être d'effondrement a des conséquences intéressantes pour la théorie des représentations de $\mathcal{V}^k(\mathfrak{g})$ et de son quotient simple $\mathcal{L}_k(\mathfrak{g})$ [AKMF+20, AFP22]. C'est aussi lié à des coincidences entre des réalisations par des algèbres vertex de théories conformes des champs en dimension 4 par la dualité 4d/2d.

En guise de dernière motivation, mentionnons qu'en type A, on a conjecturé que les réductions hamiltoniennes quantiques correspondant à des éléments nilpotents en équerre permettent de reconstruire n'importe quelle W-algèbre affine (Conjecture 6.3.2.1). Pour appuyer cette conjecture, nous en démontrons l'analogue pour les tranches de Slodowy (Théorème 6.3.2.3). Si la réduction par étapes est vérifiée dans la catégorie de Kazhdan–Lusztig, on doit pouvoir imiter la démonstration dans le cadre non-commutatif.

1.2. Introduction (in English)

Affine W-algebras form a family of graded vertex algebras constructed by quantum Hamiltonian reduction. In this thesis, we study a sufficient condition on a pair of affine W-algebras to ensure that one of them can be reconstructed as the quantum Hamiltonian reduction of the other one. This property is called *Hamiltonian reduction by stages*, by analogy with symplectic geometry [MMO+07, Mor15a, Mor15b, GJ24]. Our main motivations for such results are: establishing connections between categories of modules of various W-algebras, constructing embeddings between W-algebras or isomorphisms between their simple quotients.

Each affine W-algebra is related to some affine Poisson varieties, the corresponding Slodowy slice, and to some associative unital algebra, called finite W-algebra. The finite W-algebra is also a quantisation of the Slodowy slice. Our strategy is first to prove reduction by stages for a pair of Slodowy slices and lift this result for the corresponding finite and affine W-algebras by using relevant filtrations on them. This thesis is mainly based on two articles written by the author in collaboration with Genra [GJ24, GJ25].

1.2.1. Affine W-algebras

Vertex algebras are algebraic structures that were first axiomatised by Borcherds [Bor86] to prove the Moonshine Conjecture [FLM89, Bor92]. They are noncommutative and nonassociative differential algebras, but the lack of commutativity and associativity is controlled by the Borcherds identities. They provide a rigorous mathematical framework to define the chiral part of a two-dimensional conformal field theory [BPZ84]. Virasoro algebras and affine Kac–Moody algebras are examples of vertex algebras.

Affine W-algebras form an important family of vertex algebras obtained as quantum Hamiltonian reductions of affine Kac–Moody algebras. They were introduced in theoretical physics by Belavin, Polyakov and Zamolodchikov as generalisations of the Virasoro symmetries in conformal field theories of dimension 2 [BPZ84, Zam85]. Affine W-algebras generalise both Virasoro Lie algebras and affine Kac–Moody Lie algebras, but because of the nonlinearity in the relations between their generators, they are not generated by Lie algebras in general [BS93]. Affine W-algebras play also a role in the study of four-dimensional conformal field theory by the 4d/2d-*duality* [BLL+15, Ara19, Ara18, SXY24], in the *AGT correspondence* [SV13] or the *quantum geometric Langlands program* [Fre07, AF19].

The affine W-algebra $\mathcal{W}^k(\mathfrak{g}, f)$ is a vertex algebra defined from the data of a finite-dimensional simple complex Lie algebra $\mathfrak{g}$, a nilpotent element f in $\mathfrak{g}$ and a complex number k. If $f = 0$, then the corresponding affine W-algebra is the *universal affine vertex algebra* $\mathcal{V}^k(\mathfrak{g})$ associated with $\mathfrak{g}$ and k, that is a vertex algebra analogue of the affine Kac–Moody algebra $\widehat{\mathfrak{g}}^k$. The universal affine vertex algebra was introduced by I. Frenkel and Zhu [FZ92]. In general, $\mathcal{W}^k(\mathfrak{g}, f)$ is constructed by applying a BRST cohomology functor, denoted by H^0_f, to $\mathcal{V}^k(\mathfrak{g})$:

$$\mathcal{W}^k(\mathfrak{g}, f) \coloneqq \mathrm{H}^0_f(\mathcal{V}^k(\mathfrak{g})).$$

The *BRST (Becchi–Rouet–Stora–Tyutin) cohomology* was developed by Kostant and Sternberg to compute Hamiltonian reduction in the noncommutative setting [KS87], formalising the BRST quantisation methods from theoretical physics. Recall that given an action of a Lie group on a symplectic manifold and a moment map, one can perform Hamiltonian reduction, which consists in taking the quotient of the zero-fibre of the moment map by the action of the group. Under good assumptions, this quotient has a natural symplectic manifold structure [MW74]. BRST cohomology, applied to the ring of functions of the manifold to be reduced, computes the ring of functions of the reduced manifold.

The BRST construction of $\mathcal{W}^k(\mathfrak{g}, f)$ is due to Feigin and E. Frenkel [FF90] when f is a regular nilpotent element (meaning that its adjoint orbit is dense in the nilpotent cone) and by Kac, Roan and Wakimoto in general [KRW03, KW04]. They are the universal affine enveloping algebras of nonlinear (conformal) Lie algebras [DSK05]: that means it is finitely freely generated with nonlinear relations between the generators. In general, these generators and relations are very difficult to compute. It has been done when f is a minimal nilpotent element [KW04] or for small rank examples [CL18, Fas22, Fas25]. If $\mathfrak{g}$ is $\mathfrak{sl}_2$ and f is nonzero, then $\mathscr{W}^k(\mathfrak{sl}_2, f)$ is a Virasoro vertex algebra whenever $k+2 \neq 0$.

1.2.2. Reduction by stages

Given the action of a Lie group on a symplectic manifold, when the group contains a closed normal subgroup, the Hamiltonian reduction can be performed in two stages. One can first take the Hamiltonian reduction by this normal subgroup (first stage). Then, one can consider the induced action of the quotient of the two groups on the reduced symplectic manifold and perform Hamiltonian reduction again (second stage). This procedure is called *Hamiltonian reduction by stages* [MMO+07]. Under good assumptions, the Hamiltonian reduction by the ambient group and the reduction by stages produce the same symplectic manifold, up to a natural isomorphism.

In the setting of affine W-algebras, this construction has a natural vertex algebra analogue. Let f_1, f_2 be two nilpotent elements in $\mathfrak{g}$ and denote their adjoint orbits by $\mathbf{O}_1, \mathbf{O}_2$. Assume the inclusion $\overline{\mathbf{O}_1} \subseteq \overline{\mathbf{O}_2}$ of their Zariski closures and assume that the element $f_0 \coloneqq f_2 - f_1$ is nilpotent. We say that *reduction by stages* holds for the corresponding pair of affine W-algebras if there exists a BRST cohomology functor $\mathrm{H}^0_{f_0}$ that can be applied to the first W-algebra to get the second one, up to a natural isomorphism of vertex algebras:

$$\mathrm{H}^0_{f_0}(\mathscr{W}^k(\mathfrak{g}, f_1)) \cong \mathscr{W}^k(\mathfrak{g}, f_2).$$

In other words, we have the commutative triangle:

$$\begin{tikzcd}
& \mathscr{V}^k(\mathfrak{g}) \arrow[dl, mapsto, "\mathrm{H}^0_{f_1}"'] \arrow[dr, mapsto, "\mathrm{H}^0_{f_2}"] & \\
\mathscr{W}^k(\mathfrak{g}, f_1) \arrow[rr, mapsto, "\mathrm{H}^0_{f_0}"'] & & \mathscr{W}^k(\mathfrak{g}, f_2).
\end{tikzcd}$$

Reduction by stages was studied by Madsen and Ragoucy in the theoretical physics paper [MR97], when the simple Lie algebra $\mathfrak{g}$ is $\mathfrak{sl}_n$ and the nilpotent elements f_1, f_2 correspond to *hook-type partitions*, that is to say partitions of the form $(a, 1^{n-a})$ for $1 \leqslant a \leqslant n$. Their approach consists in proving the isomorphism

$$\mathrm{H}^0_{f_0}(\mathrm{H}^0_{f_1}(\mathscr{V}^k(\mathfrak{g}))) \cong \mathrm{H}^0_{f_2}(\mathscr{V}^k(\mathfrak{g}))$$

by noticing that the right-hand side cohomology is the total cohomology of a double cochain complex, and the left hand-side is the second page of the associated *spectral sequence*. We will generalise this approach.

Recently, reductions by stages of affine W-algebras (also called *partial reductions*) were intensively studied by using the free-field realisations of affine W-algebras developed in [Gen20]. In general, this approach allows to study a reduction by stages with its associated *inverse reduction*. Inverse reduction consists in "inverting" the BRST cohomology functor H^0_f in the sense of constructing a vertex algebra embedding

$$\mathscr{V}^k(\mathfrak{g}) \longrightarrow \mathscr{W}^k(\mathfrak{g}, f) \otimes_{\mathbf{C}} \mathscr{D}_{\mathrm{ch}},$$

where $\mathscr{D}_{\mathrm{ch}}$ denotes the vertex algebra of chiral differential operators on an affine variety of the form $\mathbf{A}^{n_1} \times \mathbf{G}_{\mathrm{m}}^{n_2}$. These embeddings are useful tools for the representation theory of affine W-algebras [Ada19, AKR21, ACG24]. This can be generalised to reduction by stages. In this context, an inverse reduction is an embedding of the form

$$\mathscr{W}^k(\mathfrak{g}, f_1) \longrightarrow \mathscr{W}^k(\mathfrak{g}, f_2) \otimes_{\mathbf{C}} \mathscr{D}_{\mathrm{ch}}.$$

(Inverse) reductions by stages were studied in [Feh23, Feh24] for type A and hook-type nilpotent elements; in [FN23] for type B and subregular, regular nilpotent elements; in [CFLN24] for type A in small ranks; in [FFFN24] for $\mathfrak{g} = \mathfrak{sl}_4$ and any ordered pair of nilpotent orbits; and in [FKN24] for $\mathfrak{g}$ a classical type and H_{f_0} being a Virasoro-type reduction. According to [But23, Section 5], inverse reductions by stages may be obtained by using the description of affine W-algebras in type A as vertex algebras associated with a divisor in a Calabi–Yau threefold [But23, Theorem 5.9].

1.2.3. Quantisation techniques

Given a noncommutative algebra with a filtration, it often happens that the associated graded algebra is commutative, with a Poisson bracket inherited from the nonzero bracket on the filtered algebra. Then the graded algebra is the algebra of functions on an affine Poisson variety (or more generally a scheme) with an action of the multiplicative group $\mathbf{G}_{\mathrm{m}}$ [BPW16]. One says that the noncommutative algebra is a *quantisation* of the affine Poisson scheme. A great idea of geometric representation theory is that the filtered noncommutative algebra and its representation theory can be better understood by studying the geometric properties of this Poisson scheme.

A classical example of quantisation is the ring of differential operators on a complex manifold, that quantises the cotangent space of this manifold. More generally, any D-module is related to the geometry of a subvariety of the cotangent space, called the characteristic variety of this D-module [Bor87, Kas03]. If $\mathscr{D}(\mathbf{A}^n)$ is the algebra of algebraic differential operators on the affine space $\mathbf{A}^n$, it is filtered by the order of the differential operators and the graded algebra $\operatorname{gr}\mathscr{D}(\mathbf{A}^n)$ is a commutative Poisson algebra that is isomorphic to the algebra of functions on the cotangent space $\mathrm{T}^*\mathbf{A}^n$.

Another example, more related to the motivations of this thesis, is the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of a finite-dimensional complex Lie algebra $\mathfrak{g}$. By the Poincaré–Birkhoff–Witt Theorem, this algebra is filtered and quantises the dual space $\mathfrak{g}^*$ equipped with its Kirillov–Kostant Poisson structure. If $\mathfrak{g}$ is simple, by the work of Joseph, the irreducible highest weight representations of $\mathfrak{g}$ are quantisations of nilpotent orbits closures in $\mathfrak{g}^*$ [Jos85].

It was established by Li that any vertex algebra is equipped with a natural filtration such that the associated graded algebra is a commutative differential algebra with a Poisson vertex structure [Li05]. Arakawa proved that for any affine Poisson scheme, the algebra of functions of the corresponding arc space is also equipped with a Poisson vertex structure [Ara12]. Following the ideas of Zhu, he also defined the *associated scheme* $X_{\mathcal{V}}$ of a (finitely generated) vertex algebra $\mathcal{V}$. In general, $\operatorname{gr}^{\mathrm{Li}} \mathcal{V}$ is a quotient of $\mathbf{C}[\mathrm{J}_\infty X_{\mathcal{V}}]$, the algebra of functions on the arc space of the associated scheme.

For example, for the universal affine vertex algebra $\mathcal{V}^k(\mathfrak{g})$, the associated graded algebra $\operatorname{gr}^{\mathrm{Li}} \mathcal{V}^k(\mathfrak{g})$ is isomorphic to the algebra of functions on the arc space $\mathrm{J}_\infty\mathfrak{g}^*$ of $\mathfrak{g}^*$ as a Poisson vertex algebra. Denote by $\mathcal{L}_k(\mathfrak{g})$ the unique simple graded quotient of $\mathcal{V}^k(\mathfrak{g})$, called the *simple affine vertex algebra*. If the complex number k is *admissible* (see Section 6.3.3 for examples in type A), then the reduced scheme associated to $X_{\mathcal{L}_k(\mathfrak{g})}$ is the closure of a nilpotent orbit depending on the level k, denoted by $\mathbf{O}_k$ [Ara15].

1.2.4. Slodowy slices and finite W-algebras

It has been proved that the Li filtration relates affine W-algebras to the arc spaces of some affine varieties, the *Slodowy slices*, that were introduced by Slodowy to study singularities of closures of nilpotent orbits in a simple Lie algebra $\mathfrak{g}$ [Slo06]. To any nilpotent element f, one can associate a Slodowy slice S_f that is an affine subspace of $\mathfrak{g}^*$ that is transverse to any coadjoint orbits intersected by the slice. This transversality implies that S_f inherits a Poisson structure from the symplectic structures on the orbits.

Gan and Ginzburg proved that the Poisson structure on S_f can also be constructed by Hamiltonian reduction of $\mathfrak{g}^*$ [GG02]. The acting group N is a unipotent group whose Lie algebra $\mathfrak{n}$ is a subalgebra of the positive part of a grading $\mathfrak{g} = \bigoplus_{\delta\in\mathbf{Z}} \mathfrak{g}_\delta$ that is *good* for f in the sense of [EK05, BG07]. Their construction is a reinterpretation and a continuation of Premets's results, who constructed the Poisson structure on Slodowy slices as induced by the bracket of the corresponding finite W-algebra [Pre02]. These results generalised the ones obtained by Kostant for a regular nilpotent element [Kos78] and by Lynch for even nilpotent elements [Lyn79].

The corresponding *finite W-algebra* $\mathcal{U}(\mathfrak{g}, f)$ was first studied by Kostant for a regular nilpotent element f [Kos78], and then Premet gave a general construction by quantum Hamiltonian reduction [Pre02]. Premet also introduced a filtration such that $\operatorname{gr} \mathcal{U}(\mathfrak{g}, f)$ is a commutative algebra isomorphic to $\mathbf{C}[S_f]$.

Figure 1.2

$$\begin{tikzcd}[column sep=large, row sep=large]
\mathscr{V}^k(\mathfrak{g}) \arrow[r, mapsto, "\mathrm{gr}^{\mathrm{Li}}"] \arrow[d, mapsto, "\mathrm{Zhu}"'] & \mathbf{C}[\mathrm{J}_\infty \mathfrak{g}^*] \arrow[l, dashed, bend right] \arrow[d, mapsto, "\mathrm{Zhu}"] \\
\mathscr{U}(\mathfrak{g}) \arrow[r, mapsto, "\mathrm{gr}"'] & \mathbf{C}[\mathfrak{g}^*] \arrow[l, dashed, bend left]
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=large, row sep=large]
\mathscr{W}^k(\mathfrak{g}, f) \arrow[r, mapsto, "\mathrm{gr}^{\mathrm{Li}}"] \arrow[d, mapsto, "\mathrm{Zhu}"'] & \mathbf{C}[\mathrm{J}_\infty S_f] \arrow[l, dashed, bend right] \arrow[d, mapsto, "\mathrm{Zhu}"] \\
\mathscr{U}(\mathfrak{g}, f) \arrow[r, mapsto, "\mathrm{gr}"'] & \mathbf{C}[S_f] \arrow[l, dashed, bend left]
\end{tikzcd}$$

Gan and Ginzburg proved that this isomorphism is compatible with Poisson structures on both sides, so the finite W-algebra quantises the Slodowy slice. This was generalised to the setting of good gradings by Brundan and Goodwin in [BG07], and by Sadaka to admissible gradings [Sad16].

The Slodowy slice is related to the affine W-algebra because $\mathrm{gr}^{\mathrm{Li}}\, \mathscr{W}^k(\mathfrak{g}, f)$ and $\mathbf{C}[\mathrm{J}_\infty S_f]$ are isomorphic as Poisson vertex algebras [DSK06, Ara15]. This relation has been used a lot to study the structure and representation theory of the affine W-algebra, see [Ara15, AM18, AvEM24, AM25]. Affine and finite W-algebras are related by the Zhu algebra construction [Ara07, DSK06]. We summarise all these algebras and their relations in Figure 1.2, where the dashed arrow represent quantisations.

The problem of reduction by stages for Slodowy slices and finite W-algebras was first studied by Morgan in his PhD thesis [Mor15a, Mor15b]. Let f_1, f_2 be two nilpotent elements in $\mathfrak{g}$ and denote by $\mathbf{O}_1, \mathbf{O}_2$ their adjoint orbits in $\mathfrak{g}$. Assume the inclusion $\overline{\mathbf{O}_1} \subseteq \overline{\mathbf{O}_2}$ and that the element $f_0 \coloneqq f_2 - f_1$ is nilpotent. Morgan conjectured that the following triangle is commutative [Mor15b, Objective 3.6]:

$$\begin{tikzcd}
 & \mathscr{U}(\mathfrak{g}) \arrow[dl, mapsto, "\mathrm{H}^0_{f_1}"'] \arrow[dr, mapsto, "\mathrm{H}^0_{f_2}"] & \\
\mathscr{U}(\mathfrak{g}, f_1) \arrow[rr, mapsto, "\mathrm{H}^0_{f_0}"'] & & \mathscr{U}(\mathfrak{g}, f_2),
\end{tikzcd}$$

and he conjectured the analogue result for Slodowy slices.

In fact, Morgan found out that stronger conditions should be assumed on the nilpotent elements f_1, f_2 to construct reduction by stages. In type A, for any pair of nilpotent elements whose orbits are adjacent for the closure ordering, he provided a general construction of a group N_0, with a Hamiltonian action on the Slodowy slice S_{f_1}, satisfying these conditions. He conjectured that the resulting Hamiltonian reduction is isomorphic to the Slodowy slice S_{f_2}, and he proved it for a particular case (subregular and regular orbits) and for more examples in small ranks: see [Mor15b, Conjecture 3.13] and the development written just after. Let us mention that reduction by stages also appears in the context of slices in the affine Grassmaniann [KPW22], that are known to be related to Slodowy slices in type A under the Mirković–Vybornov isomorphism [MV07].

1.2.5. Contributions of this thesis

Let $\mathfrak{g}$ be a simple Lie agebra and $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. For $i = 1, 2$, let H_i be in $\mathfrak{h}$ such that the associated grading

$$\mathfrak{g} = \bigoplus_{\delta \in \mathbf{Z}} \mathfrak{g}_\delta^{(i)}, \quad \text{where} \quad \mathfrak{g}_\delta^{(i)} \coloneqq \{x \in \mathfrak{g} \mid [H_i, x] = \delta x\},$$

is a good grading (Definition 2.2.1.3) for the nilpotent element f_i.

Since $[H_1, H_2] = 0$, one gets a bigrading

$$\mathfrak{g} = \bigoplus_{\delta_1, \delta_2 \in \mathbf{Z}} \mathfrak{g}_{\delta_1, \delta_2} \quad \text{where} \quad \mathfrak{g}_{\delta_1, \delta_2} \coloneqq \mathfrak{g}_{\delta_1}^{(1)} \cap \mathfrak{g}_{\delta_2}^{(2)}.$$

Set $f_0 \coloneqq f_2 - f_1$. Our sufficient conditions for reduction by stages are defined by:

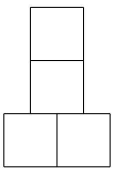
$$(\bigstar) \qquad \begin{cases} \mathfrak{g}_{\geqslant 2}^{(1)} \subseteq \mathfrak{g}_{\geqslant 1}^{(2)} \subseteq \mathfrak{g}_{\geqslant 0}^{(1)}, \quad \mathfrak{g}_1^{(1)} \subseteq \bigoplus_{\delta=0}^{2} \mathfrak{g}_{1,\delta}, \quad \mathfrak{g}_1^{(2)} \subseteq \bigoplus_{\delta=0}^{2} \mathfrak{g}_{\delta,1}, \\ f_0 \in \mathfrak{g}_{0,-2}. \end{cases}$$

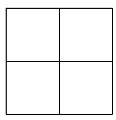
See Table 1.2 for examples for which Conditions ($\bigstar$) hold. We also give the following example as a guideline.

Example (Examples 2.3.1.5 and 6.2.1.5). Take $\mathfrak{g} = \mathfrak{sl}_4$. Consider the nilpotent elements

$$f_1 \coloneqq \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ 1 & & & 0 \end{pmatrix} \quad \text{and} \quad f_2 \coloneqq \begin{pmatrix} 0 & & & \\ & 0 & & \\ & 1 & 0 & \\ 1 & & & 0 \end{pmatrix}$$

respectively in the orbits associated to the following Young diagrams:





Then Conditions ($\bigstar$) holds for f_1 and f_2.

The conditions ($\bigstar$) are introduced in [GJ25]. These conditions imply the inclusion $\overline{\mathbf{O}_1} \subseteq \overline{\mathbf{O}_2}$ of nilpotent orbits closures (Proposition 2.3.2.2). One can construct a group N_0 (Proposition 2.3.1.1), an action of N_0 on the Slodowy S_{f_1} and a moment map $\mu_0 : S_{f_1} \to \mathfrak{n}_0{}^*$, where n_0 denotes the Lie algebra of N_0, such that the following theorem holds.

Theorem 1 ([GJ24, Theorems 1 and 2], Theorem 2.3.3.3)**.** *If Conditions* ($\bigstar$) *hold, then:*

1. *the action of N_0 induces an isomorphism $N_0 \times S_{f_2} \cong {\mu_0}^{-1}(0)$ that is N_0-equivariant, where the left-hand side is equipped with the action by left multiplication on N_0,*

2. *there is a Poisson isomorphism*

$$S_{f_2} \cong {\mu_0}^{-1}(0)//N_0.$$

There is a Lie algebra embedding $\mathfrak{n}_0 \hookrightarrow \mathcal{U}(\mathfrak{g}, f_1)$ (Lemma 4.3.1.2). So it makes sense to consider the quantum Hamiltonian reduction of $\mathcal{U}(\mathfrak{g}, f_1)$ with respect to the action of N_0. By introducing filtrations on these noncommutative objects, we can use Theorem 1 to establish the following result.

Theorem 2 ([GJ24, Theorem 3], Theorem 4.3.1.3)**.** *If Conditions* ($\bigstar$) *hold, then the quantum Hamiltonian reduction of $\mathcal{U}(\mathfrak{g}, f_1)$ with respect to the action of N_0 is isomorphic to $\mathcal{U}(\mathfrak{g}, f_2)$ as algebras.*

Theorem 2 is proved by applying Theorem 1 after introducing well-chosen filtrations on $\mathcal{U}(\mathfrak{g}, f_1)$ and $\mathcal{U}(\mathfrak{g}, f_2)$. As an application, we get an analogue of the Skryabin equivalence [Skr02]. Indeed, the nilpotent element f_0 defines a character $\overline{\chi}_0$ of $\mathfrak{n}_0$, so it makes sense to define the category of Whittaker $\mathcal{U}(\mathfrak{g}, f_1)$-modules, denoted by Wh_0: they are the left $\mathcal{U}(\mathfrak{g}, f_1)$-modules V such that for all x in $\mathfrak{n}_0$, $x + \overline{\chi}_0(x)1$ acts locally nilpotently. It fits into the following equivalence of categories.

Theorem 3 ([GJ24, Theorem 4], Proposition 4.4.3.3)**.** *There is an equivalence of categories*

$$\begin{tikzcd} \mathsf{Wh}_0 \arrow[r, bend left] \arrow[r, phantom, "\simeq"] & \mathcal{U}(\mathfrak{g}, f_2)\text{-}\mathsf{Mod}. \arrow[l, bend left] \end{tikzcd}$$

In fact, we generalise the arguments of [GG02] to prove an analogue of the Skryabin equivalence in a very general setting, see Theorem 4.4.2.2.

Remark. The study of the equivalence functors in Theorem 3 has been continued by Masut in [Mas25]. She proved that this equivalence is equivariant with respect to some bi-actions of a monoidal subcategory of $\mathcal{U}(\mathfrak{g})$-$\mathsf{Mod}$ on the categories Wh_0 and $\mathcal{U}(\mathfrak{g}, f_2)$-$\mathsf{Mod}$.

Theorem 1 implies the same statement for the arc spaces of the varieties involved (Theorem 5.3.2.1). This statement has a noncommutative analogue for the affine W-algebras that quantise these arc spaces.

Theorem 4 ([GJ25, Theorem 1], Theorem 6.2.1.3)**.** *If Conditions* ($\bigstar$) *hold, then there is a BRST cochain complex* $C^{\bullet}_{f_0}(\mathcal{W}^k(\mathfrak{g}, f_1))$ *whose cohomology is isomorphic to* $\mathcal{W}^k(\mathfrak{g}, f_2)$ *as vertex algebras:*

$$\mathrm{H}^{\bullet}\big(C^{\bullet}_{f_0}(\mathcal{W}^k(\mathfrak{g}, f_1))\big) \cong \delta_{\bullet=0}\, \mathcal{W}^k(\mathfrak{g}, f_2).$$

Examples (Chapter 3). Theorem 4 holds in the cases described in Table 1.2 because Conditions ($\bigstar$) hold. By convention, the partitions indexing the nilpotent orbits in classical types are represented by nonincreasing sequences. The column "Reference" gives the first appearance in the literature of each reduction by stages.

Table 1.2: Examples of reductions by stages

$\mathfrak{g}$	f_1	f_2	Reference
type A	hook-type	hook-type	[MR97, Feh24]
type A_3	partition of 4: $(2, 1^2)$	partition of 4: $(2, 2)$	[CFLN24]
type A_{n-1}	partition of n: $(a_1, \ldots, a_{r-1}, a_r, 1^p)$	partition of n: $(a_1, \ldots, a_{r-1}, a_r + 1, 1^{p-1})$	new for $n > 3$
type B	subregular	regular	[FN23]
type C_r	partition of r: $(2^2, 1^{2r-4})$	regular	new
type G_2	Bala-Carter label $\widetilde{\mathrm{A}}_1$	regular	new

To establish Theorem 4, we prove two important results which are also of independent interest. The first one is the definition of a new BRST complex, denoted by $\widetilde{C}^{\bullet}_{f_2}(\mathcal{V}^k(\mathfrak{g}))$, to reconstruct the W-algebra $\mathcal{W}^k(\mathfrak{g}, f_2)$, with an embedding

$$C^{\bullet}_{f_0}\big(\mathcal{W}^k(\mathfrak{g}, f_1)\big) \hookrightarrow \widetilde{C}^{\bullet}_{f_2}(\mathcal{V}^k(\mathfrak{g})).$$

Therefore, one gets an induced vertex algebra homomorphism

$$\Theta : \mathrm{H}^0\big(C^{\bullet}_{f_0}(\mathcal{W}^k(\mathfrak{g}, f_1))\big) \longrightarrow \mathrm{H}^0\big(\widetilde{C}^{\bullet}_{f_2}(\mathcal{V}^k(\mathfrak{g}))\big).$$

Theorem 5 ([GJ25, Theorem 2], Theorem 6.2.3.7)**.** *The cohomology of the complex* $\widetilde{C}^{\bullet}_{f_2}(\mathcal{V}^k(\mathfrak{g}))$ *is isomorphic to the affine W-algebra* $\mathcal{W}^k(\mathfrak{g}, f_2)$*:*

$$\mathrm{H}^{\bullet}\big(\widetilde{C}^{\bullet}_{f_2}(\mathcal{V}^k(\mathfrak{g}))\big) \cong \delta_{\bullet=0}\, \mathcal{W}^k(\mathfrak{g}, f_2).$$

Remark. Theorem 5 can be compared to Sadaka's work [Sad16] because it provides new examples of constructions of affine W-algebras that are not based on a good grading, contrary to the usual construction [Pre02, GG02, BG07, KRW03].

Then we get a natural vertex algebra homomorphism

$$\Theta : \mathrm{H}^0\Big(C^\bullet_{f_0}(\mathscr{W}^k(\mathfrak{g}, f_1))\Big) \longrightarrow \mathscr{W}^k(\mathfrak{g}, f_2).$$

To show Theorem 5, that is to say to prove that the vertex algebra homomorphism Θ is an isomorphism, we need to use the Li filtration and Theorem 1. The associated graded map $\operatorname{gr}\Theta$ is an isomorphism as a consequence of the reduction by stages for the associated Slodowy slices S_{f_1}, S_{f_2} and their arc space. The Li filtration is not a complete filtration on the BRST cochain complex, so one has convergence issues. To overcome them, we use a generalisation of [AM25, Theorem 9.7].

Theorem 6 ([GJ25, Theorem 3], Theorem 5.3.5.1)**.** *Let $(C^\bullet, d)$ be a vertex algebra BRST cochain complex. Assume technical conditions, like the existence of nice gradings and some good geometric conditions, to get the vanishing of the cohomology of the associated graded cochain complex $(\operatorname{gr}^{\mathrm{Li}} C^\bullet, \operatorname{gr}^{\mathrm{Li}} d)$.*

Then the cohomology vanishes in degrees other than 0:

$$\mathrm{H}^n(C^\bullet, d) = 0 \quad \text{for} \quad n \neq 0.$$

In degree 0 there is a natural isomorphism

$$\operatorname{gr}^{\mathrm{F}} \mathrm{H}^0(C^\bullet, d) \xrightarrow{\sim} \mathrm{H}^0(\operatorname{gr}^{\mathrm{Li}} C^\bullet, \operatorname{gr}^{\mathrm{Li}} d),$$

where the filtration F on the cohomology $\mathrm{H}^0(C^\bullet, d)$ is induced by the Li filtration on the complex $(C^\bullet, d)$.

Theorem 6 holds for all the complexes mentioned above, allowing us to show Theorem 5, and then Theorem 4.

Remark. Another application of this theorem is to prove that all the possible BRST cohomology constructions of $\mathscr{W}^k(\mathfrak{g}, f)$ are equivalent: see Theorem 6.1.2.6 or the remark before Theorem 9.7 in [AM25]. This proof is inspired by [AKM15], where equivalence of definitions is proved for the $\hbar$-adic W-algebra $\mathscr{W}^k(\mathfrak{g}, f)^\wedge_\hbar$.

1.2.6. Motivations and future work

Our next purpose is to extend reduction by stages to any module in the Kazhdan–Lusztig category $\mathsf{KL}^k(\mathfrak{g})$ (Conjecture 6.3.1.2), that is to say construct an isomorphism

$$\mathrm{H}^0_{f_0} \circ \mathrm{H}^0_{f_1}(M) \cong \mathrm{H}^0_{f_2}(M)$$

for any $\mathscr{V}^k(\mathfrak{g})$-module M in $\mathsf{KL}^k(\mathfrak{g})$. Such a result must be an efficient tool to build homomorphisms between W-algebras.

It was established in [KRW03] that any affine W-algebra $\mathscr{W}^k(\mathfrak{g}, f)$ contains as a subalgebra some universal affine vertex algebra, denoted by $\mathscr{V}^\ell(\mathfrak{a})$ (see

Proposition 6.1.4.1). If reduction by stages is true in a big enough category of modules, for a nilpotent element f_0 in $\mathfrak{a}$ we can apply the functor $\mathrm{H}^0_{f_0}$ to the embedding

$$\mathscr{V}^{\ell}(\mathfrak{a}) \hookrightarrow \mathscr{W}^k(\mathfrak{g}, f),$$

and deduce an embedding of affine W-algebras (see Conjecture 6.3.3.1):

$$\mathscr{W}^{\ell}(\mathfrak{a}, f_0) \hookrightarrow \mathscr{W}^k(\mathfrak{g}, f + f_0). \tag{1.2.6.1}$$

The embedding (1.2.6.1) is important because for some particular values of k (and ℓ), it induces an isomorphism between the simple quotients of these affine W-algebras (Conjecture 6.3.3.3)

$$\mathscr{W}_{\ell}(\mathfrak{a}, f_0) \cong \mathscr{W}_k(\mathfrak{g}, f + f_0).$$

The level k is then called *collapsing*. The fact that a level k is collapsing has very interesting consequences for the representation theory of $\mathscr{V}^k(\mathfrak{g})$ and the simple quotient $\mathscr{L}_k(\mathfrak{g})$ [AKMF$^+$20, AFP22]. This is also related to coincidences between vertex algebra realisations of four-dimensional conformal field theories under the 4d/2d-duality.

To give a last motivation, let us mention that in type A, it has been conjectured that quantum Hamiltonian reduction corresponding to hook-type nilpotent elements are enough to reconstruct any affine W-algebra (Conjecture 6.3.2.1). To support this result, we prove the analogous result for Slodowy slices (Theorem 6.3.2.3). If reduction by stages holds in the Kazhdan–Lusztig category, we must be able to adapt the proof of this theorem to the noncommutative setting.

1.3. Outline

Chapters 2 to 4 are mainly based on [GJ24]. Chapters 5 and 6 are based on [GJ25].

Chapter 2

We introduce Hamiltonian reduction for affine Poisson schemes. We provide a general construction for Hamiltonian reduction by stages in this setting (Proposition 2.1.4.4). We recall the usual Gan–Ginzburg construction of Poisson structure on a Slodowy slice by Hamiltonian reduction. We study Conditions (★) and prove that they are sufficient conditions to get Theorem 1 (2.3.3.3), that is to say, reduction by stages for Slodowy slices.

Chapter 3

We describe several examples of pairs of nilpotent elements satisfying the Conditions (★). These examples are the ones summarised in Table 1.2.

Chapter 4

We introduce quantum Hamiltonian reduction for filtered associative unital algebras. In particular, we generalise the ideas of [GG02] to give a sufficient condition for the quantum Hamiltonian reduction to be the quantisation of a geometric Hamiltonian reduction, when the acting group is unipotent (Proposition 4.1.2.5 and Theorem 4.1.3.8). We provide a general construction for quantum reduction by stages (Proposition 4.1.4.2). We recall the Premet–Gan–Ginzburg quantisation of Slodowy slice by finite W-algebras. We state and prove Theorem 2 (4.3.1.3), that is to say reduction by stages for finite W-algebras. We finally provide a generalisation of the Skryabin equivalence (Theorem 4.4.2.2) to prove Theorem 3.

Chapter 5

We recall general facts about vertex algebras, Poisson vertex algebras and arc spaces of affine Poisson schemes. In particular, we introduce the vertex algebras involved in the construction of affine W-algebras. We describe BRST cohomology in three contexts: for Poisson varieties, for their arc spaces and for vertex algebras. We state vanishing results and prove Theorem 6 (5.3.5.1).

Chapter 6

We describe the different constructions of affine W-algebras and provide a proof for the equivalence of all these definitions (Theorem 6.1.2.4). We give a geometric interpretation to the Kac–Roan–Wakimoto embedding from the perspective of reduction by stages (Proposition 6.1.4.2). We prove Theorem 4 (6.2.1.3), that is to say reduction by stages for affine W-algebras, by using the analogous result for Slodowy slices. For this purpose, we provide a new definition of the second affine W-algebra in Theorem 5 (6.2.3.7). We conclude this chapter by explaining our future directions of research and some applications that are expected.

1.4. Notation and conventions

Unless otherwise stated, all objects (vector spaces, Lie algebras, algebras, schemes, varieties, algebraic groups, vertex algebras...) are defined over the field of complex numbers $\mathbf{C}$. Denote by $\mathbf{Z}$ the ring of integers. Denote by $\delta_{a=b}$ the Kronecker symbol that is equal to 1 if $a = b$ holds, 0 otherwise.

For any affine scheme $X = \operatorname{Spec} R$, denote by $\mathbf{C}[X] \coloneqq R$ its coordinate ring. If $\phi : X \to Y$ is a homomorphism of affine schemes, the comorphism is denoted by $\phi^* : \mathbf{C}[Y] \to \mathbf{C}[X]$. If R is a $\mathbf{C}$-algebra and X is a scheme, then $X(R)$ denotes the set of R-points of X, that is to say the set of unital $\mathbf{C}$-algebra homomorphisms from $\mathbf{C}[X]$ to R.

If X is a variety (reduced affine scheme of finite type), the notation $x \in X$ means that x is a $\mathbf{C}$-point of X. A linear algebraic group is an affine variety with a group scheme structure.

The affine space of dimension n is denoted by $\mathbf{A}^n \coloneqq \operatorname{Spec} \mathbf{C}[t_1, \dots, t_n]$ and is identified with its $\mathbf{C}$-point set $\mathbf{C}^n$. The multiplicative group is denoted by $\mathbf{G}_{\mathrm{m}} \coloneqq \operatorname{Spec} \mathbf{C}[t, t^{-1}]$ and is identified with its $\mathbf{C}$-point group of invertible complex numbers $\mathbf{C}^\times$.

The Lie algebra of traceless matrices of size n is denoted by $\mathfrak{sl}_n$. The elementary matrices are denoted by $E_{i,j}$, they form a basis of the vector space of square matrices Mat_n.

2 - Hamiltonian reduction by stages and Slodowy slices

2.1. Reduction by stages for affine Poisson schemes

In this section, we recall basic facts about linear algebraic groups, their Lie algebras and their actions on schemes, see for example [Mil17] for a good introduction to the subject. Inspired by [GG02], we define Hamiltonian reduction associated with a moment map in the slightly unusual case when two groups are considered (Lemma 2.1.2.5). Finally, we introduce Hamiltonian reduction by stages by adapting [MMO+07, Section 5.2] to the setting of algebraic groups acting on Poisson schemes (Lemma 2.1.4.3 and Proposition 4.1.4.2).

2.1.1. Actions of algebraic groups

Consider a *linear algebraic group* N, meaning a reduced affine group scheme of finite type over $\mathbf{C}$. Its Lie algebra $\mathfrak{n}$ is identified as the tangent space of the group N at its unit element 1, defined as the set of linear maps $x : \mathbf{C}[N] \to \mathbf{C}$ such that for any F_1, F_2 in $\mathbf{C}[N]$, the following Leibniz rule is satisfied:

$$x(F_1F_2) = x(F_1)\,F_2(1) + F_1(1)\,x(F_2).$$

In this paper, we call *(left) N-module* a vector space V that is equipped with a (right) comodule structure $\rho : V \to \mathbf{C}[N] \otimes_{\mathbf{C}} V$ over the Hopf algebra $\mathbf{C}[N]$. The subspace of invariant vectors is

$$\begin{aligned} V^N &:= \{v \in V \mid \text{for all } \rho(v) = 1 \otimes v\} \\ &= \{v \in V \mid \text{for all } g \in N,\ g \cdot v = v\}. \end{aligned}$$

Remark 2.1.1.1. Because we are working over $\mathbf{C}$ and the group is reduced of finite type, we can check the invariance by looking only at the action of the group of $\mathbf{C}$-points of N on V.

The Lie algebra $\mathfrak{n}$ of N acts on V in the following way. For any x in $\mathfrak{n}$, the map

$$\begin{aligned} x \cdot (\bullet) : V &\longrightarrow V \\ v &\longmapsto x \cdot v := (x \otimes \mathrm{id}_V) \circ \rho(v) \end{aligned}$$

defines an endomorphism of V. We get a Lie algebra homomorphism

$$\mathfrak{n} \longrightarrow \operatorname{End} V, \quad x \longmapsto x \cdot (\bullet).$$

This map is called the *infinitesimal action* of $\mathfrak{n}$ on V. If N is connected, then

$$V^N = V^{\mathfrak{n}} := \{v \in V \mid \text{for all } x \in \mathfrak{n},\ x \cdot v = 0\}.$$

For any vector v in V, set

$$N^v \coloneqq \{g \in N \mid g \cdot v = v\} \quad \text{and} \quad \mathfrak{n}^v \coloneqq \{x \in \mathfrak{n} \mid x \cdot v = 0\}.$$

Note that N^v is an algebraic subgroup of N and $\mathfrak{n}^v$ is its Lie algebra.

Consider an action $\rho : N \times X \to X$ of an affine algebraic group N on an affine scheme X. The comorphism $\rho^* : \mathbf{C}[X] \to \mathbf{C}[N] \otimes_{\mathbf{C}} \mathbf{C}[X]$ is a (left) comodule structure. By composing ρ^* with the co-inverse map $\mathbf{C}[N] \to \mathbf{C}[N]$, one gets a left action of the group N on the coordinate ring $\mathbf{C}[X]$ by algebra automorphisms. The affine scheme $X/\!/N \coloneqq \operatorname{Spec} \mathbf{C}[X]^N$ is called the *affine GIT quotient* of X modulo the action of N.

The Lie algebra $\mathfrak{n}$ of N acts by derivations on the coordinate ring $\mathbf{C}[X]$. Denote by $\operatorname{Der} \mathbf{C}[X]$ the Lie algebra of derivations on $\mathbf{C}[X]$, then the infinitesimal action is a Lie algebra homomorphism $\mathfrak{n} \to \operatorname{Der} \mathbf{C}[X]$.

2.1.2. Hamiltonian reduction with two groups

Let X be an *affine Poisson scheme*. By definition, it means that the coordinate ring $\mathbf{C}[X]$ is equipped with a $\mathbf{C}$-linear Poisson bracket $\{\bullet, \bullet\}$, that is to say, a Lie bracket on $\mathbf{C}[X]$ which is a derivation with respect to each entry of the bracket. In the following, we will encounter the following affine Poisson schemes.

Example 2.1.2.1. Let $\mathfrak{g}$ be a finite dimensional Lie algebra. The symmetric algebra $\operatorname{Sym} \mathfrak{g}$ is equipped with the unique Poisson bracket extending the Lie bracket of $\mathfrak{g}$ to the whole algebra. Hence, $\mathfrak{g}^* = \operatorname{Spec}(\operatorname{Sym} \mathfrak{g})$ is an affine Poisson variety.

Example 2.1.2.2. Assume that G is a linear algebraic group acting by the coadjoint action Ad^* on the dual space $\mathfrak{g}^*$ of its Lie algebra. Denote by X a closed G-invariant subscheme of $\mathfrak{g}^*$. Then X is a Poisson subscheme of $\mathfrak{g}^*$. In particular, any closed coadjoint orbit $\mathbb{O}$ in $\mathfrak{g}^*$ is a closed Poisson subvariety.

Example 2.1.2.3. Let (V, ω) be a symplectic vector space. The map $v \mapsto \omega(\bullet, v)$ defines a linear isomorphism $V \cong V^*$. Under this identification, $\mathbf{C}[V] \cong \operatorname{Sym} V$ is a Poisson algebra for the bracket defined by

$$\{v, w\} \coloneqq \omega(v, w), \quad v, w \in V.$$

Assume that M, N are two affine algebraic groups such that M is a normal subgroup of N, and denote by $\mathfrak{m}, \mathfrak{n}$ their respective Lie algebras. Assume the data of an N-action on X and of an N-equivariant map $\mu : X \to \mathfrak{m}^*$ such that μ is a *moment map* for the M-action on X, that is to say the comorphism $\mu^* : \mathfrak{m} \to (\mathbf{C}[X], \{\bullet, \bullet\})$ is a Lie algebra homomorphism which makes the following triangle commute:

$$\begin{tikzcd} \mathfrak{m} \arrow[rr, "\text{action}"] \arrow[dr, "\mu^*"'] & & \operatorname{Der} \mathbf{C}[X] \\ & \mathbf{C}[X]. \arrow[ur, "F \mapsto \{F, \bullet\}"'] & \end{tikzcd} \tag{2.1.2.4}$$

The N-action restricts to the scheme-theoretic fibre $\mu^{-1}(0)$ and the affine GIT quotient $\mu^{-1}(0)/\!/N$ is called *Hamiltonian reduction*. Its coordinate ring is

$$\mathbf{C}[\mu^{-1}(0)/\!/N] = (\mathbf{C}[X]/I)^N,$$

where I is the ideal spanned by the $\mu^*(x)$ for x in $\mathfrak{m}$.

Lemma 2.1.2.5. *Assume X is equipped with an algebraic action of N by Poisson automorphisms. The Hamiltonian reduction $\mu^{-1}(0)/\!/N$ is a Poisson scheme.*

More precisely, for $F_1 \bmod I, F_2 \bmod I \in (\mathbf{C}[X]/I)^N$, the formula

$$\{F_1 \bmod I, F_2 \bmod I\} \coloneqq \{F_1, F_2\} \bmod I \tag{2.1.2.6}$$

makes sense and defines a Poisson bracket on $(\mathbf{C}[X]/I)^N$.

Remark 2.1.2.7. The statement is well-known when $M = N$, see [LGPV12, Chapter 5]. If $M = N$ is connected, any algebraic action with a moment map is automatically an action by Poisson automorphism. If $M \subsetneq N$, this statement is implicitly given in [GG02].

Proof. Introduce the associative subalgebra

$$\mathscr{P} \coloneqq \{F \in \mathbf{C}[X] \mid \text{for all } g \in N,\ g \cdot F - F \in I\}.$$

In particular $\mathbf{C}[\mu^{-1}(0)/\!/N] = \mathscr{P}/I$. If we prove that $\mathscr{P}$ is a Poisson algebra and I is a Poisson ideal of $\mathscr{P}$, then $\mathscr{P}/I$ is indeed a quotient Poisson algebra.

Because N acts by Poisson automorphisms, for all g in N and all functions F_1, F_2 on X, one gets the identity $g \cdot \{F_1, F_2\} = \{g \cdot F_1, g \cdot F_2\}$. Assume that F_1 and F_2 belong to $\mathscr{P}$, so there are $x_{1,i}, x_{2,j}$ in $\mathfrak{m}$ and $F'_{1,i}, F'_{2,j}$ in $\mathbf{C}[X]$ (where i, j are indices taking finitely many values) such that

$$g \cdot F_1 = F_1 + \sum_i \mu^*(x_{1,i}) F'_{1,i} \quad \text{and} \quad g \cdot F_2 = F_2 + \sum_j \mu^*(x_{2,j}) F'_{2,j}.$$

Hence

$$\begin{aligned} g \cdot \{F_1, F_2\} = \{F_1, F_2\} + \sum_i \{\mu^*(x_{1,i}) F'_{1,i}, F_2\} + \sum_j \{F_1, \mu^*(x_{2,j}) F'_{2,j}\} \\ + \sum_{i,j} \{\mu^*(x_{1,i}) F'_{1,i}, \mu^*(x_{2,j}) F'_{2,j}\}. \end{aligned}$$

On one hand, for $\{p, q\} = \{1, 2\}$, the Poisson bracket

$$\begin{aligned} \{\mu^*(x_{p,i}) F'_{p,i}, F_q\} &= \{\mu^*(x_{p,i}), F_q\} F'_{p,i} + \mu^*(x_{p,i}) \{F'_{p,i}, F_q\} \\ &= (x_{p,i} \cdot F_q) F'_{p,i} + \mu^*(x_{p,i}) \{F'_{p,i}, F_q\} \end{aligned}$$

belongs to the ideal I because $x_{p,i} \cdot F_q$ belongs to I. One the other hand,

$$\begin{aligned}\{\mu^*(x_{1,i})F'_{1,i}, \mu^*(x_{2,j})F'_{2,j}\} = \\ &\mu^*([x_{1,i}, x_{2,j}])F'_{1,i}F'_{2,j} + F'_{1,i}\mu^*(x_{2,j})\{\mu^*(x_{1,i}), F'_{2,j}\} \\ &\quad + \mu^*(x_{1,i})F'_{2,j}\{F'_{1,i}, \mu^*(x_{2,j})\} + \mu^*(x_{1,i})\mu^*(x_{2,j})\{F'_{1,i}, F'_{2,j}\}\end{aligned}$$

belongs to I. Therefore, $\mathscr{P}$ is stable by the Poisson bracket.

Let x be in $\mathfrak{m}$, A be in $\mathbf{C}[X]$ and F be in $\mathscr{P}$. Then

$$\{\mu^*(x)A, F\} = \mu^*(x)\{A, F\} + \{\mu^*(x), F\}A = \mu^*(x)\{A, F\} + (x \cdot F)A$$

belongs to the ideal I because $x \cdot F$ is in I. So I is Poisson normalized by the algebra $\mathscr{P}$. □

2.1.3. Hamiltonian reduction for any coadjoint orbit

Take M, N, X and $\mu : X \to \mathfrak{m}^*$ as before. Denote by $\mathbb{O}$ the N-orbit of an element χ in $\mathfrak{m}^*$, and assume that this orbit is a closed subvariety. Denote by $\mu^{-1}(\mathbb{O})$ the scheme-theoretic fibre, which is an N-invariant affine subscheme of X. The affine scheme $\mu^{-1}(\mathbb{O})//N$ is Poisson and is called the *Hamiltonian reduction* of X with respect to the action of N, the moment map μ and the orbit $\mathbb{O}$.

Equivalently, one can consider the product of Poisson varieties $X \times \mathbb{O}^-$, where $\mathbb{O}^-$ denotes the orbit of $-\chi$. This Poisson variety is equipped with the diagonal N-action and the $\mathbb{O}$-twisted moment map

$$\mu_{\mathbb{O}} : X \times \mathbb{O}^- \longrightarrow \mathfrak{m}^*, \quad (x, \xi) \longmapsto \mu(x) + \xi.$$

The natural projection $X \times \mathbb{O}^- \to X$ induces a Poisson isomorphism

$$\mu_{\mathbb{O}}{}^{-1}(0)//N \cong \mu^{-1}(\mathbb{O})//N.$$

So, up to this $\mathbb{O}$-twist, we can always assume that the Hamiltonian reduction is realized with respect to the trivial orbit $\{0\}$. This is a very classical construction, see [CdS01, Section 24.4].

2.1.4. Hamiltonian reduction by stages

Consider a linear algebraic group N_2 such that there is a semidirect product decomposition

$$N_2 = N_1 \rtimes N_0,$$

where N_1, N_0 are two closed subgroups of N_2 and N_1 is a normal subgroup. In particular, N_1 acts on N_2 by right multiplication and N_0 is isomorphic to the following affine GIT quotient as a group:

$$N_0 \cong N_2//N_1.$$

So, the quotient map $N_2 \to N_0$ is N_1-invariant.

Lemma 2.1.4.1. *1. Let V be an N_2-module. Then V^{N_1} is a submodule and there is the equality*

$$\left(V^{N_1}\right)^{N_0} = V^{N_2}.$$

2. Let X be an affine scheme with an algebraic action of N_2. The affine GIT quotient $X//N_1$ has an induced N_0-action and there is a natural isomorphism

$$(X//N_1)//N_0 \cong X//N_2.$$

Proof. Denote the comodule structure by $\rho : V \to \mathbf{C}[N_2] \otimes_{\mathbf{C}} V$. According to [Mil17, Lemma 5.15], the normality of N_1 in N_2 implies the inclusion

$$\rho(V^{N_1}) \subseteq \mathbf{C}[N_2] \otimes_{\mathbf{C}} V^{N_1},$$

so we get an induced action of N_2 on V^{N_1}.

Let us show that

$$\left(\mathbf{C}[X]^{N_1}\right)^{N_0} = \mathbf{C}[X]^{N_2}.$$

The right-to-left inclusion is clear. For the other inclusion, take v in $\left(V^{N_1}\right)^{N_0}$ and g in N_2. Denote by g_0 the image of g in the quotient N_0. By noticing that $g \cdot v = g_0 \cdot v = v$, the other inclusion is proved.

Denote by $\rho : N_2 \times X \to X$ the action map that is given on the coordinate rings by a co-action $\rho^* : \mathbf{C}[X] \to \mathbf{C}[N_2] \otimes_{\mathbf{C}} \mathbf{C}[X]$. Applying the first point to $V := \mathbf{C}[X]$ proves the second one. □

There is a well-defined coadjoint action of N_2 on $\mathfrak{n}_1{}^*$ because N_1 is normal in N_2. The restriction map $\mathfrak{n}_2{}^* \twoheadrightarrow \mathfrak{n}_1{}^*$ is a Poisson homomorphism and N_2-equivariant.

Lemma 2.1.4.2. *The elements of the orthogonal subset $\mathfrak{n}_1{}^\perp$ in $\mathfrak{n}_2{}^*$ are fixed under the coadjoint action of N_1.*

Proof. For any a in N_2 and g in N_1, the product $aga^{-1}g^{-1} = aga^{-1}g^{-1}aa^{-1}$ belongs to N_1 by normality of N_1 in N_2. By differentiating, one gets that, for any x in $\mathfrak{n}_2$, $x - \mathrm{Ad}(g)x$ belongs to $\mathfrak{n}_1$. If ξ is in $\mathfrak{n}_1{}^\perp$, we deduce that

$$\mathrm{Ad}^*(g)\xi(x) = \xi(\mathrm{Ad}(g^{-1})x) = \xi(\mathrm{Ad}(g^{-1})x - x) + \xi(x) = \xi(x),$$

hence $\mathrm{Ad}^*(g)\xi = \xi$. □

Let X be an affine Poisson scheme equipped with an action of N_2 by Poisson automorphisms and an N_2-equivariant moment map $\mu_2 : X \to \mathfrak{n}_2{}^*$. The composition of the restriction map $\mathfrak{n}_2{}^* \twoheadrightarrow \mathfrak{n}_1{}^*$ with μ_2, denoted by

$$\mu_1 : X \longrightarrow \mathfrak{n}_1{}^*, \quad x \longmapsto \mu_2(x)|_{\mathfrak{n}_1},$$

is a moment map of the N_1-action obtained by restriction and is N_2-equivariant.

For $i = 1, 2$, denote by I_i the ideal of $\mathbf{C}[X]$ spanned by the ${\mu_i}^*(x)$ for x in $\mathfrak{n}_i$. Whence, one has

$$\mathbf{C}[{\mu_i}^{-1}(0)] = \mathbf{C}[X]/I_i.$$

Lemma 2.1.4.3. *The induced action of N_0 on ${\mu_1}^{-1}(0)//N_1$ has a moment map $\mu_0 : {\mu_1}^{-1}(0)//N_1 \to {\mathfrak{n}_0}^*$ induced by the restriction of μ_2 to ${\mu_1}^{-1}(0)$.*

Proof. Because of the semidirect product decomposition $\mathfrak{n}_2 = \mathfrak{n}_1 \rtimes \mathfrak{n}_0$, the inclusion $\mathfrak{n}_0 \subseteq \mathfrak{n}_2$ induces an isomorphism of Lie algebras $\mathfrak{n}_0 \cong \mathfrak{n}_2/\mathfrak{n}_1$. There is also an restriction map ${\mathfrak{n}_2}^* \twoheadrightarrow {\mathfrak{n}_0}^*$ that restricts to an isomorphism ${\mathfrak{n}_1}^\perp \cong {\mathfrak{n}_0}^*$. In a nutshell, we get the linear N_0-equivariant isomorphisms

$$(\mathfrak{n}_2/\mathfrak{n}_1)^* \cong {\mathfrak{n}_1}^\perp \cong {\mathfrak{n}_0}^*.$$

The orthogonal subset ${\mathfrak{n}_1}^\perp \cong {\mathfrak{n}_0}^*$ in ${\mathfrak{n}_2}^*$ is the fibre of the zero subspace under the restriction map ${\mathfrak{n}_2}^* \twoheadrightarrow {\mathfrak{n}_1}^*$. Hence μ_2 restricts to a well-defined map $\mu_2 : {\mu_1}^{-1}(0) \to {\mathfrak{n}_1}^\perp$ that is N_1-invariant because of Lemma 2.2.3.4, so we get an induced map $\mu_0 : {\mu_1}^{-1}(0)//N_1 \to {\mathfrak{n}_1}^\perp \cong {\mathfrak{n}_0}^*$ fitting in the following commutative diagram:

$$\begin{tikzcd}
X \arrow[r, "\mu_2"] \arrow[rr, bend left, "\mu_1"] & {\mathfrak{n}_2}^* \arrow[r, two heads] & {\mathfrak{n}_1}^* \\
{\mu_1}^{-1}(0) \arrow[u, hook] \arrow[r, "\mu_2"] \arrow[d] & {\mathfrak{n}_1}^\perp \arrow[u, hook] \arrow[r, two heads] \arrow[d, "\wr"] & \{0\} \arrow[u, hook] \\
{\mu_1}^{-1}(0)//N_1 \arrow[r, "\mu_0"] & {\mathfrak{n}_0}^*.
\end{tikzcd}$$

The map μ_0 is N_0-equivariant because $\mu^{-1}(0) \to \mu^{-1}(0)//N_1$ and μ_2 are N_0-equivariant. It is a moment map by construction.

On the coordinate ring side, one has for any x in $\mathfrak{n}_0$:

$${\mu_0}^*(x) = {\mu_2}^*(x) \bmod I_1.$$

Take F in $\mathbf{C}[X]$ such that $F \bmod I_1$ belongs to $(\mathbf{C}[X]/I_1)^{N_1}$. One has

$$x \cdot (F \bmod I_1) = (x \cdot F) \bmod I_1$$

because $\mu^{-1}(0) \hookrightarrow X$ is N_0-equivariant and

$$(x \cdot F) \bmod I_1 = \{{\mu_2}^*(x), F\} \bmod I_1$$

because μ_2 is a moment map. Hence, we get

$$x \cdot (F \bmod I_1) = \{{\mu_2}^*(x) \bmod I_1, F \bmod I_1\} = \{{\mu_0}^*(x), F \bmod I_1\}$$

by definition of the Poisson bracket of the Hamiltonian reduction. We get that μ_0 is indeed a moment map because this relation is equivalent to a commutative triangle as in (2.1.2.4). $\square$

Proposition 2.1.4.4. *The inclusion ${\mu_2}^{-1}(0) \hookrightarrow {\mu_1}^{-1}(0)$ induces a well defined N_0-equivariant map*

$$ {\mu_2}^{-1}(0)//N_1 \longrightarrow {\mu_0}^{-1}(0). \tag{2.1.4.5} $$

It induces a map between the Hamiltonian reductions:

$$ {\mu_2}^{-1}(0)//N_2 \longrightarrow {\mu_0}^{-1}(0)//N_0. \tag{2.1.4.6} $$

If (2.1.4.5) *is an isomorphism, then so is* (2.1.4.6).

Proof. The inclusion ${\mu_2}^{-1}(0) \hookrightarrow {\mu_1}^{-1}(0)$ is clear by definition of μ_1, it is N_2-equivariant and one has the commutative diagram

$$\begin{tikzcd}
{\mu_2}^{-1}(0) \arrow[r] \arrow[d, hook] & \{0\} \arrow[d, hook] \\
{\mu_1}^{-1}(0) \arrow[r, "\mu_2"] \arrow[d] & {\mathfrak{n}_0}^* \\
{\mu_1}^{-1}(0)//N_1, \arrow[ur, "\mu_0"] &
\end{tikzcd}$$

where the top square is a fibred product. By universal properties of fibred product, we get a map

$$ {\mu_2}^{-1}(0) \longrightarrow {\mu_0}^{-1}(0) $$

that is N_1-invariant, so there is an induced N_0-equivariant map (2.1.4.5). Taking the quotient by N_0 and using Lemma 2.1.4.1, it gives the map (2.1.4.6). The last part of the statement is straightforward. $\square$

Denote by I_0 the ideal of $\mathbf{C}[{\mu_1}^{-1}(0)//N_1] = \big(\mathbf{C}[X]/I_1\big)^{N_1}$ spanned by the ${\mu_0}^*(x)$ for x in $\mathfrak{n}_0$. Hence, one has

$$ \mathbf{C}[{\mu_0}^{-1}(0)] = \big(\mathbf{C}[X]/I_1\big)^{N_1}/I_0. $$

Corollary 2.1.4.7. *The reduction by stages map* (2.1.4.6) *corresponds to the comorphism:*

$$ \Big(\big(\mathbf{C}[X]/I_1\big)^{N_1}/I_0\Big)^{N_0} \longrightarrow \big(\mathbf{C}[X]/I_2\big)^{N_2} $$
$$ (F \bmod I_1) \bmod I_0 \longmapsto F \bmod I_2. $$

In particular, it is clearly a Poisson homomorphism.

Proof. The first part of the statement is a reformulation of Proposition 2.1.4.4 in the algebra setting. The second part follows from the definition of the Poisson bracket of the Hamiltonian reduction (2.1.2.6). $\square$

The construction of this section is summarized in Figure 2.1.

Figure 2.1: Reduction by stages map

$$
\begin{tikzcd}
{\mu_2}^{-1}(0) \arrow[r, hook] \arrow[d] & {\mu_1}^{-1}(0) \arrow[d] \\
{\mu_2}^{-1}(0)//N_1 \arrow[r] \arrow[dr] \arrow[dd] & {\mu_1}^{-1}(0)//N_1 \\
 & {\mu_0}^{-1}(0) \arrow[u, hook] \arrow[d] \\
{\mu_2}^{-1}(0)//N_2 \arrow[r] & {\mu_0}^{-1}(0)//N_0
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathbf{C}[X]/I_1 \arrow[r, two heads] & \mathbf{C}[X]/I_2 \\
\big(\mathbf{C}[X]/I_1\big)^{N_1} \arrow[u, hook] \arrow[r] \arrow[d, two heads] & \big(\mathbf{C}[X]/I_2\big)^{N_1} \arrow[u, hook] \\
\big(\mathbf{C}[X]/I_1\big)^{N_1}/I_0 \arrow[ur] & \\
\Big(\big(\mathbf{C}[X]/I_1\big)^{N_1}/I_0\Big)^{N_0} \arrow[u, hook] \arrow[r] & \big(\mathbf{C}[X]/I_2\big)^{N_2} \arrow[uu, hook]
\end{tikzcd}
$$

2.2. Poisson structure of Slodowy slices

In this section, we recall the construction of the Poisson structure on the Slodowy slice S_f by Hamiltonian reduction, deduced from the Kostant–Gan–Ginzburg isomorphism (Theorem 2.2.3.3), according to [GG02]. The version of the construction given here is slightly modified: Gan and Ginzburg considered Hamiltonian reductions with respect to a character, we consider a nontrivial coadjoint orbit (Proposition 2.2.2.2) as suggested in [DSK06]. The construction of [GG02] can be done in several way. In the perspective of the study of the corresponding affine W-algebra, we recall that all the constructions lead to the same Poisson structure on S_f (Corollary 2.2.5.1), that is due to [GG02, BG07]. Finally, we use reduction by stages to construct a natural action of an algebraic group $G^\natural$ on S_f and a moment map (Proposition 2.2.6.1), giving a new interpretation of a result of Premet [Pre07].

2.2.1. $\mathfrak{sl}_2$-triple and good grading

Let $\mathfrak{g}$ be a simple finite-dimensional complex Lie algebra and G be a connected algebraic group whose Lie algebra is $\mathfrak{g}$. The group G acts on itself by conjugation and on $\mathfrak{g}$ by the adjoint action Ad. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Denote by $(\bullet|\bullet)$ the non-degenerate symmetric bilinear form on $\mathfrak{g}$ given by

$$(\bullet|\bullet) \coloneqq \frac{1}{2h^\vee}\kappa_{\mathfrak{g}},$$

where $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$ and $\kappa_{\mathfrak{g}}$ is the Killing form of the Lie algebra $\mathfrak{g}$. This form is invariant by any automorphism of the Lie algebra $\mathfrak{g}$.

The dual space $\mathfrak{g}^*$ is a Poisson variety and the coadjoint action Ad^* of G on $\mathfrak{g}^*$ is by Poisson automorphisms. This bilinear form $(\bullet|\bullet)$ induces a G-equivariant isomorphism $\mathfrak{g} \cong \mathfrak{g}^*$.

Consider a nilpotent (adjoint) orbit $\mathbf{O}$ in $\mathfrak{g}$. According to the Jacobson–Morozov Theorem [CM93, Section 3.3], for any nilpotent element f in $\mathbf{O}$ there exist a nilpotent element e in $\mathfrak{g}$ and a semisimple element h in $\mathfrak{g}$ such that

$$[h,e]=2e, \quad [h,f]=-2f \quad \text{and} \quad [e,f]=h.$$

A triple (e,h,f) satisfying this bracket relations is called an $\mathfrak{sl}_2$-*triple*.

Remark 2.2.1.1. For any $\mathfrak{sl}_2$-triple (e,h,f) in $\mathfrak{g}$, e and f are always in the same nilpotent orbit. Indeed, since SL_2 is simply connected, the $\mathfrak{sl}_2$-triple induces an algebraic group homomorphism $\gamma : \mathrm{SL}_2 \to G$. Because

$$\left(\begin{smallmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right),$$

where $\mathrm{i}^2=-1$, we get

$$\mathrm{Ad}\left(\gamma\left(\left(\begin{smallmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{smallmatrix}\right)\right)\right)f = e.$$

The following property is well-known.

Proposition 2.2.1.2. *The following direct sum decompositions hold:*

$$\mathfrak{g} = \mathfrak{g}^f \oplus [\mathfrak{g},e] = [\mathfrak{g},f] \oplus \mathfrak{g}^e.$$

Proof. The adjoint action of the Lie subalgebra spanned by the $\mathfrak{sl}_2$-triple makes $\mathfrak{g}$ a representation of $\mathfrak{sl}_2$. By decomposing $\mathfrak{g}$ as a direct sum of simple representations of $\mathfrak{sl}_2$ and by using the explicit description of the simple representations of $\mathfrak{sl}_2$ (see [Hum72, Section 7]), the decomposition is clear. $\square$

Using the action of G on triples, one can choose an $\mathfrak{sl}_2$-triple (e,h,f) such that h belongs to $\mathfrak{h}$. We fix such a choice for the rest of the section.

A *Lie algebra* $\mathbf{Z}$*-grading* of $\mathfrak{g}$ is a $\mathbf{Z}$-grading $\mathfrak{g} = \bigoplus_{\delta\in\mathbf{Z}} \mathfrak{g}_\delta$ as vector space such that for all δ, δ' in $\mathbf{Z}$, one has the inclusion $[\mathfrak{g}_\delta, \mathfrak{g}_{\delta'}] \subseteq \mathfrak{g}_{\delta+\delta'}$. Such a grading corresponds to an action of $\mathbf{G}_{\mathrm{m}}$ on $\mathfrak{g}$ by Lie algebra automorphisms:

the action of t in $\mathbf{G}_{\mathrm{m}}$ on x in $\mathfrak{g}_\delta$ is given by $t \cdot x := t^\delta x$. The bilinear form $(\bullet|\bullet)$ is $\mathbf{G}_{\mathrm{m}}$-invariant, so it is a perfect pairing between $\mathfrak{g}_\delta$ and $\mathfrak{g}_{-\delta}$ for any δ in $\mathbf{Z}$, and if $\delta + \delta' \neq 0$, then $\mathfrak{g}_\delta$ and $\mathfrak{g}_{\delta'}$ are orthogonal.

Because the Lie algebra is simple, any Lie algebra grading corresponds to the eigenspaces of the adjoint action of a semisimple element H in $\mathfrak{g}$ [Hum72, Theorem 5.3], that is to say of the form

$$\mathfrak{g} = \bigoplus_{\delta \in \mathbf{C}} \mathfrak{g}_\delta^{(H)}, \quad \text{where} \quad \mathfrak{g}_\delta^{(H)} := \{x \in \mathfrak{g} \mid [H, x] = \delta x\}.$$

Definition 2.2.1.3 ([EK05]). *A Lie algebra* $\mathbf{Z}$*-grading* $\mathfrak{g} = \bigoplus_{\delta \in \mathbf{Z}} \mathfrak{g}_\delta$ *is said* good *for a nilpotent element* f *if* f *belongs to* $\mathfrak{g}_{-2}$ *and if the map induced by the adjoint action of* f*,* $\mathrm{ad}(f) : \mathfrak{g}_\delta \to \mathfrak{g}_{\delta-2}$*, is injective for* $\delta \geqslant 1$ *and surjective for* $\delta \leqslant 1$*.*

Example 2.2.1.4. By using the structure of $\mathfrak{sl}_2$-module on $\mathfrak{g}$ and the explicit description of the simple $\mathfrak{sl}_2$-modules, one can prove that the action of h induces a good grading, called the *Dynkin grading*.

Properties and classification of good gradings are studied in [EK05, BG07]. We recall some useful facts. We choose a good grading $\mathfrak{g} = \bigoplus_{\delta \in \mathbf{Z}} \mathfrak{g}_\delta$ such that the grading is induced by the adjoint action of an element H in the Cartan subalgebra $\mathfrak{h}$. According to [BG07, Lemma 19], it is automatic that e belongs to $\mathfrak{g}_2$ and h to $\mathfrak{g}_0$. When it happens, we say that the good grading and the $\mathfrak{sl}_2$-triple are *compatible*.

Proposition 2.2.1.5 ([EK05]). *For a good grading* $\mathfrak{g} = \bigoplus_{\delta \in \mathbf{Z}} \mathfrak{g}_\delta$ *for* f *and a compatible* $\mathfrak{sl}_2$*-triple* (e, h, f)*, we have the following properties:*

1. *in the decompositions* $\mathfrak{g} = \mathfrak{g}^f \oplus [\mathfrak{g}, e] = [\mathfrak{g}, f] \oplus \mathfrak{g}^e$ *given in Proposition 2.2.1.2, the direct summands are graded subspaces,*
2. *the inclusions* $\mathfrak{g}^f \subseteq \mathfrak{g}_{\leqslant 0}$ *and* $\mathfrak{g}^e \subseteq \mathfrak{g}_{\geqslant 0}$ *hold,*
3. *the map* $\mathrm{ad}(e) : \mathfrak{g}_\delta \to \mathfrak{g}_{\delta+2}$ *is injective for* $\delta \leqslant -1$*, surjective for* $\delta \geqslant -1$*,*
4. *the following sequences are exact:*

$$0 \longrightarrow \mathfrak{g}^f \longrightarrow \mathfrak{g}_{\leqslant 1} \xrightarrow{\mathrm{ad}(f)} \mathfrak{g}_{\leqslant -1} \longrightarrow 0,$$
$$0 \longrightarrow \mathfrak{g}^e \longrightarrow \mathfrak{g}_{\geqslant -1} \xrightarrow{\mathrm{ad}(e)} \mathfrak{g}_{\geqslant 1} \longrightarrow 0.$$

Proof. Property 1 is true because e and f are homogeneous elements.

The first inclusion is by definition of a good grading for f and we deduce the inclusion $\mathfrak{g}_{\geqslant 1} \subseteq [\mathfrak{g}, e]$. It is well-known that $\mathfrak{g}^e$ is the orthogonal of $[\mathfrak{g}, e]$

with respect to the bilinear form $(\bullet|\bullet)$ and $\mathfrak{g}_{\geqslant 0}$ is the orthogonal of $\mathfrak{g}_{\geqslant 1}$, hence we get the desired inclusion: $\mathfrak{g}^e \subseteq \mathfrak{g}_{\geqslant 0}$. Property 2 is proved.

The injectivity part follows from the second inclusion. To prove the surjectivity, it is equivalent to prove the injectivity of the dual map $(\mathfrak{g}_{\delta+2})^* \to {\mathfrak{g}_\delta}^*$ for $\delta \geqslant -1$. Because the bilinear form $(\bullet|\bullet)$ is a perfect pairing between $\mathfrak{g}_{-\delta-2}$ and $\mathfrak{g}_{\delta+2}$, it is equivalent to show that the linear map

$$\mathfrak{g}_{-\delta-2} \longrightarrow {\mathfrak{g}_\delta}^*, \quad x \longmapsto (x|[e,\bullet])$$

is injective. If x in $\mathfrak{g}_{-\delta-2}$ is such that $(x|[e,y]) = 0$ for all y in $\mathfrak{g}_\delta$, then the identity $(x|[e,y]) = ([x,e]|y)$, the injectivity of $\mathrm{ad}(e)$ and the perfect pairing imply the conclusion $x = 0$ and Property 3.

The first sequence is exact by definition of the good grading. The second one is exact because of Property 3. $\square$

2.2.2. Hamiltonian action

Set $\chi := (f|\bullet)$ the linear form on $\mathfrak{g}$ given by the scalar product by f. The homogeneous subspace $\mathfrak{g}_1$ is equipped with the skewsymmetric form

$$\omega(v,w) := \chi([v,w]), \quad \text{for} \quad v, w \in \mathfrak{g}_1.$$

Lemma 2.2.2.1. *The form ω is symplectic.*

Proof. Because the grading is good, the homomorphism $\mathrm{ad}(f) : \mathfrak{g}_1 \to \mathfrak{g}_{-1}$ is bijective. Using the fact that the bilinear form is a perfect pairing between $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$, we can conclude. $\square$

Let $\mathfrak{l}$ be an isotropic subspace of $\mathfrak{g}_1$ and denote by $\mathfrak{l}^{\perp,\omega}$ its orthogonal space with respect to ω. The quotient $\mathfrak{l}^{\perp,\omega}/\mathfrak{l}$ is equipped with a skewsymmetric form induced by ω which is a symplectic form, also denoted by ω.

Define the following nilpotent subalgebra of $\mathfrak{g}$:

$$\mathfrak{n}_\mathfrak{l} := \mathfrak{l}^{\perp,\omega} \oplus \mathfrak{g}_{\geqslant 2}.$$

It is the Lie algebra of a unique unipotent subgroup $N_\mathfrak{l}$ of G. The group $N_\mathfrak{l}$ acts on $\mathfrak{g}^*$ by restriction of the coadjoint action, and the restriction map gives a moment map

$$\pi_\mathfrak{l} : \mathfrak{g}^* \twoheadrightarrow {\mathfrak{n}_\mathfrak{l}}^*, \quad \xi \longmapsto \xi|_{\mathfrak{n}_\mathfrak{l}}.$$

Denote by $\mathbb{O}_\mathfrak{l}^- := -\mathrm{Ad}^*(N_\mathfrak{l})\overline{\chi}_\mathfrak{l}$ the opposite coadjoint orbit. The fibre of $\mathbb{O}_\mathfrak{l}^-$ under $\pi_\mathfrak{l}$ is equal to

$${\pi_\mathfrak{l}}^{-1}(\mathbb{O}_\mathfrak{l}^-) = -\chi + \mathrm{ad}^*(\mathfrak{k})\chi \oplus {\mathfrak{n}_l}^\perp$$

where $\mathfrak{k}$ is a subspace of $\mathfrak{l}^{\perp,\omega}$ such that $\mathfrak{l}^{\perp,\omega} = \mathfrak{k} \oplus \mathfrak{l}$.

The linear form χ restricts to a linear form on $\mathfrak{n}_\mathfrak{l}$, denoted by $\overline{\chi}_\mathfrak{l}$. Denote by $\mathbb{O}_\mathfrak{l} := \mathrm{Ad}^*(N_\mathfrak{l})\overline{\chi}_\mathfrak{l}$ its coadjoint orbit in ${\mathfrak{n}_\mathfrak{l}}^*$, which is a smooth symplectic variety. The following lemma is stated without proof in [DSK06, Section 0.4].

Proposition 2.2.2.2. *The map*

$$\sigma_{\mathfrak{l}} : \mathfrak{l}^{\perp,\omega}/\mathfrak{l} \longrightarrow \mathbb{O}_{\mathfrak{l}}, \quad (v \bmod \mathfrak{l}) \longmapsto \overline{\chi}_{\mathfrak{l}} + \mathrm{ad}^*(v)\overline{\chi}_{\mathfrak{l}},$$

is well-defined and is a symplectic isomorphism. In particular, $\mathbb{O}_{\mathfrak{l}}$ is an affine subspace of ${\mathfrak{n}_{\mathfrak{l}}}^*$.

Proof. The action of the group $N_{\mathfrak{l}}$ is given by exponentiation of its Lie algebra action, so

$$\mathrm{Ad}^*(N_{\mathfrak{l}})\overline{\chi}_{\mathfrak{l}} = \Big\{ \sum_{k\geqslant 0} \frac{1}{k!}\,\mathrm{ad}^*(v)^k\overline{\chi}_{\mathfrak{l}} \mid v \in \mathfrak{n}_{\mathfrak{l}} \Big\}.$$

Because of the good grading, $\mathrm{ad}^*(v)^k\overline{\chi}_{\mathfrak{l}} = 0$ for $k \geqslant 2$. Indeed $\mathrm{ad}(v)^k f$ belongs to $\mathfrak{g}_{k-2} \subseteq \mathfrak{g}_{\geqslant 0}$, which is orthogonal to $\mathfrak{n}_{\mathfrak{l}} \subseteq \mathfrak{g}_{\geqslant 1}$ by the bilinear form $(\bullet|\bullet)$. Hence,

$$\mathrm{Ad}^*(N_{\mathfrak{l}})\overline{\chi}_{\mathfrak{l}} = \{\overline{\chi}_{\mathfrak{l}} + \mathrm{ad}^*(v)\overline{\chi}_{\mathfrak{l}} \mid v \in \mathfrak{n}_{\mathfrak{l}}\}.$$

For degree reason, one can check that $\mathrm{ad}^*(v)^k\overline{\chi}_{\mathfrak{l}} = 0$ for v in $\mathfrak{g}_{\geqslant 2}$. So we get a surjective affine map

$$\mathfrak{l}^{\perp,\omega} \twoheadrightarrow \mathbb{O}_{\mathfrak{l}}, \quad v \longmapsto \overline{\chi}_{\mathfrak{l}} + \mathrm{ad}^*(v)\overline{\chi}_{\mathfrak{l}},$$

and the fact that $\mathfrak{l}$ is isotropic implies the isomorphism $\mathfrak{l}^{\perp,\omega}/\mathfrak{l} \cong \mathbb{O}_{\mathfrak{l}}$. □

2.2.3. Hamiltonian reduction

We want to compute the Hamiltonian reduction of $\mathfrak{g}^*$ with respect to the action of $N_{\mathfrak{l}}$ and the coadjoint orbit $\mathbb{O}_{\mathfrak{l}}^-$. To do so, we use the *Slodowy slice* associated with the $\mathfrak{sl}_2$-triple (e, h, f), that is the affine subspace of $\mathfrak{g}^*$ defined as

$$S_f \coloneqq -\chi + [\mathfrak{g}, e]^{\perp}, \quad \text{where} \quad [\mathfrak{g}, e]^{\perp} \coloneqq \{\xi \in \mathfrak{g}^* \mid \xi([\mathfrak{g}, e]) = 0\}.$$

Lemma 2.2.3.1. *The Slodowy slice S_f is contained in this fibre ${\pi_{\mathfrak{l}}}^{-1}(\mathbb{O}_{\mathfrak{l}}^-)$.*

Proof. In the proof of Proposition 2.2.1.5, we noticed the inclusion $\mathfrak{g}_{\geqslant 1} \subseteq [\mathfrak{g}, e]$ and, because $\mathfrak{n}_{\mathfrak{l}}$ is contained in $\mathfrak{g}_{\geqslant 1}$, we deduce that $[\mathfrak{g}, e]^{\perp}$ is included in ${\mathfrak{n}_{\mathfrak{l}}}^{\perp}$. The lemma follows. □

Remark 2.2.3.2. In [Slo06, Section 7.4], the Slodowy slice is defined as the affine subspace $f + \mathfrak{g}^e$ of $\mathfrak{g}$. When one is interested on the Poisson structure of these objects, it is usual to deal with the image of this slice under the isomorphism $\mathfrak{g} \cong \mathfrak{g}^*$, that is equal to $\chi + [\mathfrak{g}, e]^{\perp}$ (because of the equality $[\mathfrak{g}, e]^{\perp,(\bullet|\bullet)} = \mathfrak{g}^e$) [GG02, Los10].

To follow the conventions of [KRW03, KW04, DSK06], we work with $-\chi$ instead of χ. Indeed, the $\mathfrak{sl}_2$-triples (e, h, f) and $(-e, h, -f)$ have the same central element, so they are G-conjugate according to the Mal'cev Theorem [CM93, Theorem 3.4.12].

Theorem 2.2.3.3 ([GG02, Lemma 2.1])**.** *The action map*

$$\alpha_{\mathfrak{l}} : N_{\mathfrak{l}} \times S_f \xrightarrow{\sim} {\pi_{\mathfrak{l}}}^{-1}(\mathbb{O}_{\mathfrak{l}}^-), \quad (g, \xi) \longmapsto \mathrm{Ad}^*(g)\xi,$$

is well-defined and is an algebraic $N_{\mathfrak{l}}$-equivariant isomorphism, where the left-hand side is equipped with the action by left multiplication on $N_{\mathfrak{l}}$. Hence, the Slodowy slice is isomorphic to the Hamiltonian reduction:

$$S_f \cong {\pi_{\mathfrak{l}}}^{-1}(\mathbb{O}_{\mathfrak{l}}^-)//N_{\mathfrak{l}}.$$

The rest of the section is devoted to the proof of the theorem, that will rely on the following technical fact.

Lemma 2.2.3.4 ([Slo06, Lemma 8.1.1])**.** *Let V, W be two finite-dimensional $\mathbf{G}_{\mathrm{m}}$-modules. Let $f : V \to W$ be a $\mathbf{G}_{\mathrm{m}}$-equivariant algebraic variety homomorphism. Assume that*

1. *the $\mathbf{G}_{\mathrm{m}}$-weights of V and W are positive, that implies that 0 are the only fixed points on both sides and $f(0) = 0$,*

2. *the differential map $\mathrm{d}f_0 : V \to W$ is a linear isomorphism.*

Then $f : V \to W$ is an algebraic isomorphism.

Denote by G_{ad} the *adjoint group* of $\mathfrak{g}$, that is to say the connected component of the identity in the group of Lie algebra automorphisms of $\mathfrak{g}$. The good grading induces a $\mathbf{G}_{\mathrm{m}}$-action on $\mathfrak{g}$ by Lie algebra automorphisms, corresponding to a group homomorphism $\gamma : \mathbf{G}_{\mathrm{m}} \to G_{\mathrm{ad}}$. The weight-spaces of the $\mathbf{G}_{\mathrm{m}}$-action $\mathrm{Ad}^* \circ \gamma$ defines a grading $\mathfrak{g}^* = \bigoplus_{\delta \in \mathbf{Z}} \mathfrak{g}^*_\delta$ of the dual space $\mathfrak{g}^*$ and χ belongs to $\mathfrak{g}^*_{-2}$ because f is in $\mathfrak{g}_{-2}$.

Set $\rho : \mathbf{G}_{\mathrm{m}} \to \mathrm{Aut}_{\mathbf{C}}(\mathfrak{g}^*)$ the action defined for all $t \in \mathbf{G}_{\mathrm{m}}$ and ξ in $\mathfrak{g}^*$ by

$$\rho(t)\xi \coloneqq t^2 \,\mathrm{Ad}^*(\gamma(t))\xi. \tag{2.2.3.5}$$

In particular if ξ is in $\mathfrak{g}^*_\delta$, for δ in $\mathbf{Z}$, then $\rho(t)\xi \coloneqq t^{2+\delta}\xi$. The linear form χ is fixed by ρ. As a consequence, the Slodowy slice S_f and the fibre $\pi^{-1}(\mathbb{O}_l^-)$ are ρ-stable affine subspaces of $\mathfrak{g}^*$.

Because $N_{\mathfrak{l}}$ is unipotent, there is an algebraic isomorphism $\exp : \mathfrak{n}_l \xrightarrow{\sim} N_{\mathfrak{l}}$ such that for all x, y in $\mathfrak{n}_l$, one has the equality

$$\mathrm{Ad}(\exp(x))y = \exp(\mathrm{ad}(x))y \coloneqq \sum_{p=0}^{\infty} \frac{1}{p!}\,\mathrm{ad}(x)^p y,$$

where the right-hand side sum is finite by nilpotency.

Lemma 2.2.3.6. *There is a $\mathbf{G}_{\mathrm{m}}$-action on $N_{\mathfrak{l}}$ by group automorphisms such that the exponential map $\exp : \mathfrak{n}_l \to N_{\mathfrak{l}}$ is $\mathbf{G}_{\mathrm{m}}$-equivariant. The coadjoint action of the group $N_{\mathfrak{l}}$*

$$\alpha_{\mathfrak{l}} : N_{\mathfrak{l}} \times \mathfrak{g}^* \longrightarrow \mathfrak{g}^*, \quad (g, \xi) \longmapsto \mathrm{Ad}^*(g)\xi$$

that is $\mathbf{G}_{\mathrm{m}}$-equivariant.

This lemma is proved in [GG02, Lemma 2.1] in the case of a Dynkin grading, by considering the map $\mathrm{SL}_2 \to G$ induced by the $\mathfrak{sl}_2$-triple. We adapt the proof for any good grading.

Proof. Above we defined a group homomorphism $\gamma : \mathbf{G}_{\mathrm{m}} \to G_{\mathrm{ad}}$. By using the fact that G_{ad} is isomorphic to the quotient of G by its centre, the conjugation action of G on itself induces an action of G_{ad} on G by group automorphisms and then a $\mathbf{G}_{\mathrm{m}}$-action on G. Because $N_{\mathfrak{l}}$ is connected and $\mathfrak{n}_{\mathfrak{l}}$ is $\mathbf{G}_{\mathrm{m}}$-stable, $N_{\mathfrak{l}}$ is stable by the $\mathbf{G}_{\mathrm{m}}$-action.

For the equivariance part, it is enough to prove that the map

$$G \times \mathfrak{g}^* \longrightarrow \mathfrak{g}^*, \quad (g, \xi) \longmapsto \mathrm{Ad}^*(g)\xi$$

is $\mathbf{G}_{\mathrm{m}}$-equivariant. It is a consequence of the following computation

$$\begin{aligned}\mathrm{Ad}^*(\gamma(t))(\mathrm{Ad}^*(g)\xi) &= \mathrm{Ad}^*(\gamma(t))\,\mathrm{Ad}^*(g)\,\mathrm{Ad}^*(\gamma(t^{-1}))\,\mathrm{Ad}^*(\gamma(t))\xi \\ &= \mathrm{Ad}^*(\gamma(t)g\gamma(t^{-1}))\,\mathrm{Ad}^*(\gamma(t))\xi.\end{aligned}$$

□

We are ready to prove the main theorem.

Proof of Theorem 2.2.3.3. The map $\alpha_{\mathfrak{l}} : N_{\mathfrak{l}} \times S_f \to {\pi_{\mathfrak{l}}}^{-1}(\mathbb{O}_{\mathfrak{l}}^-)$ is well-defined because the right-hand side is stable by the $N_{\mathfrak{l}}$-action and because of the inclusion $S_f \subseteq {\pi_{\mathfrak{l}}}^{-1}(\mathbb{O}_{\mathfrak{l}}^-)$.

We consider the vector spaces $\mathfrak{n}_{\mathfrak{l}}$ equipped with the $\mathbf{G}_{\mathrm{m}}$-action induced by the good grading and $[\mathfrak{g}, e]^{\perp}$, $\mathrm{ad}^*(\mathfrak{k})\chi \oplus {\mathfrak{n}_l}^{\perp}$ equipped with the restriction of the $\mathbf{G}_{\mathrm{m}}$-action ρ. The isomorphism

$$\begin{aligned}\mathfrak{n}_{\mathfrak{l}} \times [\mathfrak{g}, e]^{\perp} &\longrightarrow N_l \times S_f, \quad (x, \xi) \longmapsto (\exp(x), -\chi + \xi) \\ \mathrm{ad}^*(\mathfrak{k})\chi \oplus {\mathfrak{n}_l}^{\perp} &\longrightarrow {\pi_{\mathfrak{l}}}^{-1}(\mathbb{O}_{\mathfrak{l}}^-), \quad \xi \longmapsto -\chi + \xi\end{aligned}$$

are $\mathbf{G}_{\mathrm{m}}$-equivariant because χ is fixed by ρ. After to conjugating $\alpha_{\mathfrak{l}}$ by these isomorphisms, we could apply Lemma 2.2.3.4 if the following properties hold:

1. the $\mathbf{G}_{\mathrm{m}}$-weights of $\mathfrak{n}_{\mathfrak{l}}$, $[\mathfrak{g}, e]^{\perp}$ and $\mathrm{ad}^*(\mathfrak{k})\chi \oplus {\mathfrak{n}_l}^{\perp}$ are positive,
2. the differential map $\mathrm{d}(\alpha_{\mathfrak{l}})_{(1,-\chi)} : \mathfrak{n}_{\mathfrak{l}} \oplus [\mathfrak{g}, e]^{\perp} \to \mathrm{ad}^*(\mathfrak{k})\chi \oplus {\mathfrak{n}_l}^{\perp}$ is a linear isomorphism.

The Lie algebra $\mathfrak{n}_\mathfrak{l}$ is contained in $\mathfrak{g}_{\geqslant 1}$ so its weights are clearly positive. The isomorphism $\mathfrak{g} \cong \mathfrak{g}^*$ maps respectively $\mathfrak{g}^e$ and $[f,\mathfrak{k}] \oplus {\mathfrak{n}_l}^{\perp,\omega}$ to $[\mathfrak{g},e]^\perp$ and $\mathrm{ad}^*(\mathfrak{k})\chi \oplus {\mathfrak{n}_l}^\perp$. According to Proposition 2.2.1.5, we have the inclusion of $\mathfrak{g}^e$ in $\mathfrak{g}_{\geqslant 0}$, so the ρ-weights of $[\mathfrak{g},e]^\perp$ are greater than 2. Because $\mathfrak{k}$ is a subspace of $\mathfrak{g}_1$, and because the inclusion $\mathfrak{g}_{\geqslant 2} \subseteq \mathfrak{n}_\mathfrak{l}$ implies ${\mathfrak{n}_\mathfrak{l}}^{\perp,\omega} \subseteq (\mathfrak{g}_{\geqslant 2})^{\perp,(\bullet|\bullet)} = \mathfrak{g}_{\geqslant -1}$, the space $[f,\mathfrak{k}] \oplus {\mathfrak{n}_l}^{\perp,\omega}$ is contained in $\mathfrak{g}_{\geqslant -1}$, so the ρ-weights of $\mathrm{ad}^*(\mathfrak{k})\chi \oplus {\mathfrak{n}_l}^\perp$ are greater than 1. Therefore, Property 1 holds.

The differential map is given for x in $\mathfrak{n}_\mathfrak{l}$ and ξ in $[\mathfrak{g},e]^\perp$ by

$$\mathrm{d}(\alpha_\mathfrak{l})_{(1,-\chi)}(x,\xi) = -\,\mathrm{ad}^*(x)\chi + \xi.$$

Because of the good grading property, $x \mapsto -\,\mathrm{ad}^*(x)\chi$ is injective and using the transverse intersection $\mathfrak{g}^e \cap [\mathfrak{n}_\mathfrak{l}, f] \cong [\mathfrak{g},e]^\perp \cap \mathrm{ad}^*(\mathfrak{n}_l)\chi = 0$ (Proposition 2.2.1.2), we deduce that $\mathrm{d}(\alpha_\mathfrak{l})_{(1,-\chi)}$ is injective.

To prove surjectivity, we compare dimensions:

$$\begin{aligned}
\dim(\mathrm{ad}^*(\mathfrak{k})\chi \oplus {\mathfrak{n}_l}^\perp) &= \dim \mathfrak{l}^{\perp,\omega} - \dim \mathfrak{l} + \dim \mathfrak{g} - \dim \mathfrak{n}_\mathfrak{l} \\
&= \dim \mathfrak{g} - \dim \mathfrak{l} - \dim \mathfrak{g}_{\geqslant 2} \\
&= \dim \mathfrak{g}_{\geqslant -1} - \dim \mathfrak{l}
\end{aligned}$$

and

$$\dim(\mathfrak{n}_\mathfrak{l} \oplus [\mathfrak{g},e]^\perp) = \dim \mathfrak{l}^{\perp,\omega} + \dim \mathfrak{g}_{\geqslant 2} + \dim \mathfrak{g}^e,$$

hence

$$\dim(\mathrm{ad}^*(\mathfrak{k})\chi \oplus {\mathfrak{n}_l}^\perp) - \dim(\mathfrak{n}_\mathfrak{l} \oplus [\mathfrak{g},e]^\perp) = \dim \mathfrak{g}_{\geqslant -1} - \dim \mathfrak{g}^e - \dim \mathfrak{g}_{\geqslant 1},$$

and this difference is zero because of the second exact sequence of Property 4 in Proposition 2.2.1.5. Dimensions are equal so the differential is bijective and Property 2 holds.

We can apply Lemma 2.2.3.4 and the theorem is proved. □

In the following, we will use the reformulation of this construction with the twisted moment map. Consider the Poisson variety $\mathfrak{g}^* \times (\mathfrak{l}^{\perp,\omega}/\mathfrak{l})$. It has a natural diagonal action of $N_\mathfrak{l}$ given by the coadjoint action on $\mathfrak{g}^*$ and the isomorphism of $\mathfrak{l}^{\perp,\omega} \cong \mathbb{O}_\mathfrak{l}$. There is a moment map

$$\mu_\mathfrak{l} : \mathfrak{g}^* \times (\mathfrak{l}^{\perp,\omega}/\mathfrak{l}) \longrightarrow {\mathfrak{n}_\mathfrak{l}}^*, \quad (\xi, v \bmod \mathfrak{l}) \longmapsto \pi_\mathfrak{l}(\xi) + \overline{\chi}_\mathfrak{l} + \mathrm{ad}^*(v)\overline{\chi}_\mathfrak{l} \tag{2.2.3.7}$$

It is clear that the natural projection $\mathfrak{g}^* \times (\mathfrak{l}^{\perp,\omega}/\mathfrak{l}) \twoheadrightarrow \mathfrak{g}^*$ induces an $N_\mathfrak{l}$-isomorphism

$$\mu_\mathfrak{l}^{-1}(0) \cong {\pi_\mathfrak{l}}^{-1}(\mathbb{O}_\mathfrak{l}^-).$$

Taking the quotients by $N_\mathfrak{l}$, we get that the Hamiltonian reductions are Poisson isomorphic:

$$\mu_\mathfrak{l}^{-1}(0)/\!/N_\mathfrak{l} \cong {\pi_\mathfrak{l}}^{-1}(\mathbb{O}_\mathfrak{l}^-)/\!/N_\mathfrak{l}.$$

2.2.4. Functorial Hamiltonian reduction

Let X be an affine Poisson scheme equipped with an action of the group G by Poisson automorphisms and a G-equivariant moment map $\phi_X : X \to \mathfrak{g}^*$. There is an action of $N_{\mathfrak{l}}$ on X and the composition $\pi_X \coloneqq \pi_{\mathfrak{l}} \circ \phi_X : X \to {\mathfrak{n}_{\mathfrak{l}}}^*$ is a moment map. Denote by ${\phi_X}^{-1}(S_f)$ the scheme-theoretic fibre of S_f and consider ${\phi_X}^{-1}({\pi_{\mathfrak{l}}}^{-1}(\mathbb{O}_{\mathfrak{l}}^-)) = {\pi_X}^{-1}(\mathbb{O}_{\mathfrak{l}}^-)$. The following proposition is a well-known generalisation of [Gin09, Proposition 1.3.3].

Proposition 2.2.4.1. *The action of $N_{\mathfrak{l}}$ on X induces an isomorphism*

$$N_{\mathfrak{l}} \times {\phi_X}^{-1}(S_f) \cong {\pi_X}^{-1}(\mathbb{O}_{\mathfrak{l}}^-).$$

This proposition is a straightforward consequence of Theorem 2.2.3.3 and the following lemma.

Lemma 2.2.4.2. *Let Z be an affine scheme with an action of an algebraic group M. Assume that there exists a closed subscheme S of Z such that the action induces an isomorphism $M \times S \cong Z$. Then for any M-equivariant map of affine schemes $\phi : Z' \to Z$, the action map induces an isomorphism*

$$M \times \phi^{-1}(S) \cong Z',$$

where $\phi^{-1}(S)$ denote the scheme-theoretic fibre of S under ϕ.

Proof. By fibred product, the isomorphism $M \times S \cong Z$ induces an isomorphism

$$Z' \times_Z (M \times S) \cong Z',$$

where the left-hand side fibreed prdocut is defined by the map $\phi : Z' \to Z$ and the action map $M \times S \to Z$. To conclude the proof of the lemma, it is enough to prove that the action map

$$M \times \phi^{-1}(S) \coloneqq M \times (Z' \times_Z S) \longrightarrow Z' \times_Z (M \times S)$$

is an isomorphism.

Both $M \times \phi^{-1}(S), Z' \times_Z (M \times S)$ are closed subschemes of $M \times Z' \times S$ respectively defined by the equations $\phi(z) = s$ and $\phi(z) = g \cdot s$ for g in M, z in Z and s in S. Hence, the automorphism

$$M \times Z' \times S \longrightarrow M \times Z' \times S, \quad (g, z, s) \longmapsto (g, g \cdot z, s)$$

given by the action induces the isomorphism that we want. □

2.2.5. Equivalence of the constructions

The previous construction requires many choices: a nilpotent element f, a good grading defined by a semisimple element H and an isotropic subspace $\mathfrak{l}$. We want to prove that the corresponding Hamiltonian reductions are isomorphic as Poisson varieties. In the above discussion, all the objects related to $\mathfrak{l}$ were denoted with a "$\mathfrak{l}$" subscript. For $\mathfrak{l} = \{0\}$, we omit the subscript. In particular, $\mathfrak{n} = \mathfrak{g}_{\geqslant 1}$, $N = G_{\geqslant 1}$, there is an isomorphism $\mathfrak{g}_1 \cong \mathcal{O}$ and $\pi^{-1}(\mathcal{O}^-) = -\chi + (\mathfrak{g}_{\geqslant 2})^\perp$. The following corollary is an straightforward consequence of Theorem 2.2.3.3.

Corollary 2.2.5.1 ([GG02, Section 5.5])**.** *For any isotropic subspace $\mathfrak{l}$ in $\mathfrak{g}_1$, there is an inclusion ${\pi_\mathfrak{l}}^{-1}(\mathcal{O}_\mathfrak{l}^-) \subseteq \pi^{-1}(\mathcal{O}^-)$ which induces a Poisson isomorphism*

$$ {\pi_\mathfrak{l}}^{-1}(\mathcal{O}_\mathfrak{l}^-)//N_\mathfrak{l} \cong \pi^{-1}(\mathcal{O}^-)//N. $$

Let us denote by

$$ \mathscr{P}(\mathfrak{g}, f, H) \coloneqq \mathbf{C}[{\pi_\mathfrak{l}}^{-1}(\mathcal{O}_\mathfrak{l}^-)]^{N_\mathfrak{l}} $$

the Poisson algebra obtained as the coordinate ring of one of these Hamiltonians reductions, for any choice of coisotropic subspace $\mathfrak{l}$ of $\mathfrak{g}_1$. Using the vocabulary of [DSKV16], we call this Poisson algebra the *classical finite W-algebra* associated with the pair (H, f).

Corollary 2.2.5.1 is used in [BG07, Sections 4 and 5] to prove that for any nilpotent elements f, f' in the orbit $\mathbf{O}$ and for any semisimple elements H, H' defining good gradings for these nilpotent elements, there is a Poisson algebra isomorphism between the associated W-algebras:

$$ \mathscr{P}(\mathfrak{g}, f, H) \cong \mathscr{P}(\mathfrak{g}, f', H'). $$

As a consequence, the classical finite W-algebras $\mathscr{P}(\mathfrak{g}, f, H)$ constructed above only depend on the nilpotent orbit $\mathbf{O}$ which contains f. They are called *the classical finite* W-algebra associated with the orbit $\mathbf{O}$ at level k and denoted by $\mathscr{P}(\mathfrak{g}, f)$.

2.2.6. Hamiltonian action on the Slodowy slice

Let $G^\natural$ be the connected subgroup of G whose Lie algebra is

$$ \mathfrak{g}^\natural \coloneqq \mathfrak{g}^f \cap \mathfrak{g}_0. $$

In other terms, $G^\natural$ is the connected component of the neutral element in the intersection $G^f \cap G^H$.

Consider the twisted moment map defined in (2.2.3.7):

$$ \mu : \mathfrak{g}^* \times \mathfrak{g}_1 \longrightarrow \mathfrak{n}^*, \quad (\xi, v) \longmapsto \pi(\xi) + \overline{\chi} + \mathrm{ad}^*(v)\overline{\chi}, $$

for the action of $N \coloneqq G_{\geqslant 1}$. Denote by $\mathbb{O}$ the coadjoint orbit of $\overline{\chi}$ in $\mathfrak{n}^*$, where $\overline{\chi}$ denotes the restriction of χ to $\mathfrak{n}$. It gives a realisation of the Slodowy slice as Hamiltonian reduction:

$$S_f \cong \mu^{-1}(0)//N.$$

Let $\{v_i\}_{i=1}^{2s}$ be a basis of the symplectic space $\mathfrak{g}_1$ and denote by $\{v^i\}_{i=1}^{2s}$ the symplectic dual basis, these vectors correspond to elements in $\mathbf{C}[\mathfrak{g}_1] \cong \operatorname{Sym} \mathfrak{g}_1$. Denote by I the ideal of $\mathbf{C}[\mathfrak{g}^*] \otimes_{\mathbf{C}} \mathbf{C}[\mathfrak{g}_1]$ spanned by the $\mu^*(x)$ for x in $\mathfrak{n}$.

Proposition 2.2.6.1 ([Pre02, Lemmas 2.3 and 2.4])**.** *There is an action of $G^\natural$ on $\mu^{-1}(0)//N$ and a moment map $\theta^\natural : \mu^{-1}(0)//N \to (\mathfrak{g}^\natural)^*$ whose comorphism is given for x in $\mathfrak{g}^\natural$ by:*

$$(\theta^\natural)^*(x) = x \otimes 1 + \frac{1}{2}\sum_{i=1}^{2s} 1 \otimes v^i[v_i, x] \bmod I,$$

that lies in $\mathscr{P}(\mathfrak{g}, f) = \big((\mathbf{C}[\mathfrak{g}^*] \otimes_{\mathbf{C}} \mathbf{C}[\mathfrak{g}_1])/I\big)^N$.

Remark 2.2.6.2. In [Pre02, Lemmas 2.3 and 2.4], Premet constructs this action for the corresponding finite W-algebra. In [KRW03, KW04], an analogue of this action is constructed for the corresponding affine W-algebra (see Proposition 6.1.4.1). In both cases, the formula for the moment map is found by the explicit computation of the generators of $\mathbf{C}[S_f] \cong \mathscr{P}(\mathfrak{g}, f)$ that are of degree 2 with respect to the $\mathbf{G}_{\mathrm{m}}$-action given in (2.2.3.5).

We give a new proof of Proposition 2.2.6.1 that uses reduction by stages. The group $G^\natural$ normalises $N = G_{\geqslant 1}$. Define

$$N^\natural \coloneqq N \rtimes G^\natural$$

the semi-direct product of these groups. Denote by $\mathfrak{n}^\natural$ the associated Lie algebra, by $\overline{\chi}^\natural$ the restriction of χ to $\mathfrak{n}^\natural$ and by $\mathbb{O}^\natural$ its coadjoint orbit in $(\mathfrak{n}^\natural)^*$.

Lemma 2.2.6.3. *There is a symplectic isomorphism given by the map*

$$\sigma^\natural : \mathfrak{g}_1 \longrightarrow \mathbb{O}^\natural, \quad v \longmapsto \overline{\chi}^\natural + \operatorname{ad}^*(v)\overline{\chi}^\natural + \frac{1}{2}\operatorname{ad}^*(v)^2\overline{\chi}^\natural.$$

Moreover, the restriction map $(\mathfrak{n}^\natural)^* \twoheadrightarrow \mathfrak{n}^*$ induces a symplectic isomorphism between the coadjoint orbits $\mathbb{O}^\natural$ and $\mathbb{O}$:

$$\begin{tikzcd}
 & \mathbb{O}^\natural \arrow[r, hook] \arrow[d, "\wr"] & (\mathfrak{n}^\natural)^* \arrow[d, two heads] \\
\mathfrak{g}_1 \arrow[ur, "\sigma^\natural", "\sim"'] \arrow[dr, "\sim", "\sigma"'] & & \\
 & \mathbb{O} \arrow[r, hook] & \mathfrak{n}^*.
\end{tikzcd}$$

Proof. The argument is similar to Proposition 2.2.2.2, see Proposition 6.2.2.2 for details. □

Proof of Proposition 2.2.6.1. The group $N^\natural$ acts by the diagonal action on the Cartesian product $\mathfrak{g}^* \times \mathfrak{g}_1 \cong \mathfrak{g}^* \times \mathbb{O}^\natural$ and there is a moment map

$$\mu^\natural : \mathfrak{g}^* \times \mathfrak{g}_1 \longrightarrow (\mathfrak{n}^\natural)^*, \quad (\xi, v) \longmapsto \xi|_{\mathfrak{n}^\natural} + \overline{\chi}^\natural + \mathrm{ad}^*(v)\overline{\chi}^\natural + \frac{1}{2}\,\mathrm{ad}^*(v)^2\overline{\chi}^\natural.$$

The restriction of this action to the normal subgroup N corresponds to the usual moment map $\mu : \mathfrak{g}^* \times \mathfrak{g}_1 \to \mathfrak{n}^*$.

The first part of the statement is then a direct application of Proposition 2.1.4.4 to the moment map $\mu^\natural : \mathfrak{g}^* \times \mathfrak{g}_1 \to (\mathfrak{n}^\natural)^*$. The comorphism is given by this formula because $\theta^\natural$ is induced by $\mu^\natural$, whose comorphism is given for x in $\mathfrak{g}^\natural$ by:

$$(\mu^\natural)^*(x) = x \otimes 1 + \frac{1}{2}\sum_{i=1}^{2s} 1 \otimes v^i[v_i, x].$$

□

Remark 2.2.6.4. The action of $G^\natural$ is naturally defined on $\mu^{-1}(0)/\!/N$, but it is not easy to observe on the Slodowy slice S_f itself because the slice is not stable by the coadjoint action of $G^\natural$. Consider the reductive part of $G^\natural$, that corresponds the connected component of the identity in $G^f \cap G^e$ and denoted by $G^\natural_{\mathrm{red}}$. The coadjoint action of $G^\natural_{\mathrm{red}}$ stabilises S_f but this group is smaller than $G^\natural$ in general.

Example 2.2.6.5. Take $\mathfrak{g} = \mathfrak{sl}_3$ and consider the $\mathfrak{sl}_2$-triple

$$e := \begin{pmatrix} 0 & 1 & \\ & 0 & \\ & & 0 \end{pmatrix}, \quad h := \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix}, \quad \text{and} \quad f := \begin{pmatrix} 0 & & \\ 1 & 0 & \\ & & 0 \end{pmatrix}.$$

The adjoint action of

$$H := \frac{1}{3}\begin{pmatrix} 4 & & \\ & -2 & \\ & & -2 \end{pmatrix}$$

induces a good grading for f whose nonnegative parts are given by:

$$\begin{aligned} \mathfrak{g}_0 &= \mathbf{C}(E_{1,1} - E_{2,2}) \oplus \mathbf{C}(E_{2,2} - E_{3,3}) \oplus \mathbf{C}E_{2,3} \oplus \mathbf{C}E_{3,2} \\ \mathfrak{g}_1 &= \mathbf{C}E_{1,2} \oplus \mathbf{C}E_{1,3}. \end{aligned}$$

For $G = \mathrm{SL}_3$, one can check that

$$\begin{aligned} G^\natural &= \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & a & b \\ 0 & 0 & a^{-2} \end{pmatrix} \mid a \in \mathbf{C}^\times,\ b \in \mathbf{C} \right\}, \\ G^\natural_{\mathrm{red}} &= \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a^{-2} \end{pmatrix} \mid a \in \mathbf{C}^\times \right\} \cong \mathbf{G}_{\mathrm{m}}. \end{aligned}$$

2.3. Reduction by stages for Slodowy slices

In this section, we define the Conditions ($\bigstar$) and study their consequences as done in [GJ25, Section 5.1]. We prove that if a pair of nilpotent elements f_1, f_2 satisfy these conditions, then there is an inclusion of the closures of their nilpotent orbits (Proposition 2.3.2.2). We state and prove reduction by stages for Slodowy slices by following [GJ24], see Theorem 1 (2.3.3.3).

2.3.1. Sufficient condition on the good gradings

For $i = 1, 2$, let f_i be a nilpotent element in $\mathfrak{g}$ and H_i be an element in $\mathfrak{h}$ such that the grading

$$\mathfrak{g} = \bigoplus_{\delta \in \mathbf{Z}} \mathfrak{g}_\delta^{(i)}, \quad \mathfrak{g}_\delta^{(i)} \coloneqq \mathfrak{g}_\delta^{(H_i)},$$

is a good grading for f_i. The homogeneous subspace $\mathfrak{g}_1^{(i)}$ is equipped with the symplectic form $\omega_i(u,v) \coloneqq (f_i|[u,v])$ for u, v in $\mathfrak{g}_1^{(i)}$. Introduce the Lie subalgebra $\mathfrak{g}^{\natural,1} \coloneqq \mathfrak{g}_0^{(1)} \cap \mathfrak{g}^{f_1}$.

Since $[H_1, H_2] = 0$, one gets a bigrading on $\mathfrak{g}$:

$$\mathfrak{g} = \bigoplus_{\delta_1, \delta_2 \in \mathbf{Z}} \mathfrak{g}_{\delta_1,\delta_2}, \quad \text{where} \quad \mathfrak{g}_{\delta_1,\delta_2} \coloneqq \mathfrak{g}_{\delta_1}^{(1)} \cap \mathfrak{g}_{\delta_2}^{(2)}.$$

Set $f_0 \coloneqq f_2 - f_1$. Consider the following conditions:

$$(\bigstar) \qquad \begin{cases} \mathfrak{g}_{\geqslant 2}^{(1)} \subseteq \mathfrak{g}_{\geqslant 1}^{(2)} \subseteq \mathfrak{g}_{\geqslant 0}^{(1)}, \quad \mathfrak{g}_1^{(1)} \subseteq \bigoplus_{\delta=0}^{2} \mathfrak{g}_{1,\delta}, \quad \mathfrak{g}_1^{(2)} \subseteq \bigoplus_{\delta=0}^{2} \mathfrak{g}_{\delta,1}, \\ f_0 \in \mathfrak{g}_{0,-2}. \end{cases}$$

Proposition 2.3.1.1. *Assume that Conditions ($\bigstar$) hold. Then there exist an isotropic subspace $\mathfrak{l}_1$ of $\mathfrak{g}_1^{(1)}$ and an isotropic subspace $\mathfrak{l}_2$ of $\mathfrak{g}_1^{(2)}$ such that both are H_1 and H_2-stable, and the nilpotent algebras*

$$\mathfrak{n}_1 \coloneqq {\mathfrak{l}_1}^{\perp,\omega_1} \oplus \mathfrak{g}_{\geqslant 2}^{(1)} \quad \textit{and} \quad \mathfrak{n}_2 \coloneqq {\mathfrak{l}_2}^{\perp,\omega_2} \oplus \mathfrak{g}_{\geqslant 2}^{(2)}$$

satisfy the following properties:

1. *the algebra $\mathfrak{n}_1$ is an ideal of $\mathfrak{n}_2$,*
2. *there exists a subalgebra of $\mathfrak{n}_2$, denoted by $\mathfrak{n}_0$, such that it is contained in the Lie subalgebra $\mathfrak{g}^{\natural,1}$ and there is a decomposition $\mathfrak{n}_2 = \mathfrak{n}_1 \oplus \mathfrak{n}_0$.*

In the rest of this section, we assume the conditions ($\bigstar$). To prove the proposition, we start with the following lemmas.

Lemma 2.3.1.2. *The forms ω_1 and ω_2 coincide on $\mathfrak{g}_{1,1} = \mathfrak{g}_1^{(1)} \cap \mathfrak{g}_1^{(2)}$ and the subspace $\mathfrak{g}_{1,1}$ is a symplectic subspace of $\mathfrak{g}_1^{(1)}$ and of $\mathfrak{g}_1^{(2)}$.*

Proof. Take $x, y \in \mathfrak{g}_{1,1}$. Then, $[x, y] \in \mathfrak{g}_{2,2}$ is orthogonal to $f_0 \in \mathfrak{g}_{0,-2}$. Hence

$$\omega_1(x, y) = (f_1|[x, y]) = (f_2|[x, y]) = \omega_2(x, y).$$

Let us prove that ω_1 is nondegenerate on $\mathfrak{g}_{1,1}$. Because the H_1-grading is good, the adjoint action of f_1 in $\mathfrak{g}_{-2,-2}$ induces an isomorphism $\mathfrak{g}_{1,1} \cong \mathfrak{g}_{-1,-1}$. Moreover, the symmetric invariant bilinear form $(\bullet|\bullet)$ is perfect pairing between the subspaces $\mathfrak{g}_{1,1}$ and $\mathfrak{g}_{-1,-1}$, hence ω_1 is nondegenerate. $\square$

Lemma 2.3.1.3. *The subspace* $\mathfrak{l}_1 \coloneqq \mathfrak{g}_{1,2}$ *is isotropic in* $\mathfrak{g}_1^{(1)}$ *and its orthogonal is given by* ${\mathfrak{l}_1}^{\perp,\omega_1} = \mathfrak{g}_{1,1} \oplus \mathfrak{g}_{1,2}$.

Proof. Because of the hypotheses of Proposition 2.3.1.1, the vector space $\mathfrak{g}_1^{(1)}$ decomposes as $\mathfrak{g}_1^{(1)} = \mathfrak{g}_{1,0} \oplus \mathfrak{g}_{1,1} \oplus \mathfrak{g}_{1,2}$. The adjoint action of f_1 induces an isomorphism $\mathfrak{g}_{1,2} \cong \mathfrak{g}_{-1,0}$ and the bilinear form $(\bullet|\bullet)$ is a perfect pairing between the subspaces $\mathfrak{g}_{-1,0}$ and $\mathfrak{g}_{1,0}$, so $\mathfrak{g}_{1,0}$ and $\mathfrak{g}_{1,2}$ are perfectly paired by ω_1 and have the same dimension. Hence, $\mathfrak{g}_{1,0} \oplus \mathfrak{g}_{1,2}$ is a symplectic subspace of $\mathfrak{g}_1^{(1)}$ and $\mathfrak{g}_{1,2}$ is a Lagrangian subspace of this symplectic subspace.

By comparing the bidegrees, one can also see that $\omega_1(\mathfrak{g}_{1,1}, \mathfrak{g}_{1,0} \oplus \mathfrak{g}_{1,2}) = 0$. Hence, the subspace $\mathfrak{g}_{1,2}$ is isotropic and its orthogonal is $\mathfrak{g}_{1,1} \oplus \mathfrak{g}_{1,2}$. $\square$

Lemma 2.3.1.4.

1. *The subspace* $\mathfrak{g}_{0,1} \oplus \mathfrak{g}_{2,1}$ *is a symplectic subspace of* $\mathfrak{g}_1^{(2)}$.
2. *There exists a subspace* $\mathfrak{a}$ *in* $\mathfrak{g}_{0,1} \cap \mathfrak{g}^{f_1}$ *such that* $\mathfrak{a} \oplus \mathfrak{g}_{2,1}$ *is a Lagrangian subspace of* $\mathfrak{g}_{0,1} \oplus \mathfrak{g}_{2,1}$.
3. *The subspace* $\mathfrak{l}_2 \coloneqq \mathfrak{a} \oplus \mathfrak{g}_{2,1}$ *is isotropic in* $\mathfrak{g}_1^{(2)}$ *and its orthogonal is given by* ${\mathfrak{l}_2}^{\perp,\omega_2} = \mathfrak{a} \oplus \mathfrak{g}_{1,1} \oplus \mathfrak{g}_{2,1}$.

Proof. Because f_0 belongs to $\mathfrak{g}_{0,-2}$, one has the inclusion

$$[f_2, \mathfrak{g}_{1,1}] \subseteq \mathfrak{g}_{1,-1} \oplus \mathfrak{g}_{-1,-1}$$

and this space is orthogonal to $\mathfrak{g}_{0,1} \oplus \mathfrak{g}_{2,1}$ by the pairing $(\bullet|\bullet)$. Hence,

$$\omega_2(\mathfrak{g}_{0,1} \oplus \mathfrak{g}_{2,1}, \mathfrak{g}_{1,1}) = 0.$$

The symplectic form ω_2 has to be nondegenerate on $\mathfrak{g}_{0,1} \oplus \mathfrak{g}_{2,1}$, whence it is a symplectic subspace and (1) is proved.

In the same way, one has $[f_2, \mathfrak{g}_{2,1}] \subseteq \mathfrak{g}_{0,-1} \oplus \mathfrak{g}_{2,-1}$, so the subspace $\mathfrak{g}_{2,1}$ is isotropic in $\mathfrak{g}_{0,1} \oplus \mathfrak{g}_{2,1}$. It can be extended to a Lagrangian subspace of the form $\mathfrak{a} \oplus \mathfrak{g}_{2,1}$, with $\mathfrak{a} \subseteq \mathfrak{g}_{0,1}$.

We claim that $\mathfrak{a}$ is included in $\mathfrak{g}_{0,1}$ belongs to $\mathfrak{g}^{f_1}$. Any element in $\mathfrak{a}$ is of the form $x = u + v$ where $u \in \mathfrak{g}^{f_1}$ and v belongs to a complement of $\mathfrak{g}^{f_1}$ in $\mathfrak{g}$ which is H_1 and H_2-stable. Then,

$$[f_2, x] = [f_0, u] + [f_1, v] + [f_0, v],$$

where $[f_0, u] \in \mathfrak{g}_{0,-1}$, $[f_1, v] \in \mathfrak{g}_{-2,-1}$ and $[f_0, v] \in \mathfrak{g}_{0,-1}$. Since x belongs to the isotropic subspace $\mathfrak{a} \oplus \mathfrak{g}_{2,1}$, for all $y \in \mathfrak{g}_{2,1}$:

$$\omega_2(x, y) = ([f_1, v]|y) = 0.$$

Hence, $[f_1, v]$ is zero because the subspace $\mathfrak{g}_{-2,-1}$ is paired with $\mathfrak{g}_{2,1}$. Finally, we deduce that v has to be zero because it lies in a complement of $\mathfrak{g}^{f_1}$. This proves the existence of $\mathfrak{a}$ as in (2).

By construction, it is clear that $\mathfrak{a} \oplus \mathfrak{g}_{2,1}$ is isotropic in $\mathfrak{g}_1^{(2)}$ and its orthogonal is the subspace $\mathfrak{a} \oplus \mathfrak{g}_{1,1} \oplus \mathfrak{g}_{2,1}$, so (3) follows. □

Proof of Proposition 2.3.1.1. Consider the following subspaces of the Lie algebra $\mathfrak{g}$:

$$\mathfrak{n}_0 := \mathfrak{a} \oplus (\mathfrak{g}_0^{(1)} \cap \mathfrak{g}_{\geqslant 2}^{(2)}), \quad \mathfrak{n}_1 := {\mathfrak{l}_1}^{\perp,\omega_1} \oplus \mathfrak{g}_{\geqslant 2}^{(1)}, \quad \mathfrak{n}_2 := {\mathfrak{l}_2}^{\perp,\omega_2} \oplus \mathfrak{g}_{\geqslant 2}^{(2)}.$$

We have the decomposition

$$\mathfrak{g}_{\geqslant 1}^{(2)} = (\mathfrak{g}_1^{(2)} \cap \mathfrak{g}_0^{(1)}) \oplus (\mathfrak{g}_1^{(2)} \cap \mathfrak{g}_1^{(1)}) \oplus (\mathfrak{g}_1^{(2)} \cap \mathfrak{g}_2^{(1)}) \oplus (\mathfrak{g}_{\geqslant 2}^{(2)} \cap \mathfrak{g}_0^{(1)}) \oplus (\mathfrak{g}_2^{(2)} \cap \mathfrak{g}_1^{(1)}) \oplus (\mathfrak{g}_{\geqslant 2}^{(2)} \cap \mathfrak{g}_{\geqslant 2}^{(1)}),$$

whence we get the decomposition

$$\mathfrak{n}_2 = \mathfrak{a} \oplus (\mathfrak{g}_1^{(2)} \cap \mathfrak{g}_1^{(1)}) \oplus (\mathfrak{g}_1^{(2)} \cap \mathfrak{g}_2^{(1)}) \oplus (\mathfrak{g}_{\geqslant 2}^{(2)} \cap \mathfrak{g}_0^{(1)}) \oplus (\mathfrak{g}_2^{(2)} \cap \mathfrak{g}_1^{(1)}) \oplus (\mathfrak{g}_{\geqslant 2}^{(2)} \cap \mathfrak{g}_{\geqslant 2}^{(1)}).$$

We have the decomposition

$$\mathfrak{g}_{\geqslant 1}^{(1)} = (\mathfrak{g}_1^{(1)} \cap \mathfrak{g}_0^{(2)}) \oplus (\mathfrak{g}_1^{(1)} \cap \mathfrak{g}_1^{(2)}) \oplus (\mathfrak{g}_1^{(1)} \cap \mathfrak{g}_2^{(2)}) \oplus (\mathfrak{g}_2^{(1)} \cap \mathfrak{g}_1^{(2)}) \oplus (\mathfrak{g}_{\geqslant 2}^{(1)} \cap \mathfrak{g}_{\geqslant 2}^{(2)}),$$

whence we have the decomposition

$$\mathfrak{n}_1 = (\mathfrak{g}_1^{(1)} \cap \mathfrak{g}_1^{(2)}) \oplus (\mathfrak{g}_1^{(1)} \cap \mathfrak{g}_2^{(2)}) \oplus (\mathfrak{g}_2^{(1)} \cap \mathfrak{g}_1^{(2)}) \oplus (\mathfrak{g}_{\geqslant 2}^{(1)} \cap \mathfrak{g}_{\geqslant 2}^{(2)}).$$

To prove (1), we just need to prove the inclusion $[{\mathfrak{l}_1}^{\perp,\omega_1}, \mathfrak{n}_0] \subseteq {\mathfrak{l}_1}^{\perp,\omega_1}$ because the H_1-grading is a Lie grading. But this inclusion is clear since

$${\mathfrak{l}_1}^{\perp,\omega_1} = \mathfrak{g}_1^{(1)} \cap \mathfrak{g}_{\geqslant 1}^{(2)} \quad \text{and} \quad \mathfrak{n}_0 \subseteq \mathfrak{g}_0^{(1)} \cap \mathfrak{g}_{\geqslant 1}^{(2)}.$$

After comparing these decompositions, is is clear that $\mathfrak{n}_2 = \mathfrak{n}_1 \oplus \mathfrak{n}_0$. To prove (2), we just need to prove that $\mathfrak{n}_0 \subseteq \mathfrak{g}^{f_1}$. It is clear because of the inclusions

$$\mathfrak{a} \subseteq \mathfrak{g}^{f_1} \quad \text{and} \quad [f_1, \mathfrak{g}_0^{(1)} \cap \mathfrak{g}_{\geqslant 2}^{(2)}] \subseteq \mathfrak{g}_{-2}^{(1)} \cap \mathfrak{g}_{\geqslant 0}^{(2)} = 0,$$

where the last equality is a consequence of the inclusion $\mathfrak{g}_{-2}^{(1)} \subseteq \mathfrak{g}_{\leqslant -1}^{(2)}$. □

Example 2.3.1.5. Take $\mathfrak{g} = \mathfrak{sl}_4$. Consider the nilpotent elements

$$f_1 := \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ 1 & & & 0 \end{pmatrix} \quad \text{and} \quad f_2 := \begin{pmatrix} 0 & & & \\ & 0 & & \\ & 1 & 0 & \\ 1 & & & 0 \end{pmatrix}.$$

They are embedded in $\mathfrak{sl}_2$-triples (e_i, h_i, f_i), $i = 1, 2$, where

$$e_1 \coloneqq E_{1,4}, \quad e_2 \coloneqq E_{1,4} + E_{2,3}$$

and

$$h_1 \coloneqq E_{1,1} - E_{4,4}, \quad h_2 \coloneqq E_{1,1} + E_{2,2} - E_{3,3} - E_{4,4}.$$

The adjoint actions of h_1 and h_2 induce good gradings (in fact Dynkin gradings) for, respectively, f_1 and f_2. Their nonnegative parts are given by:

$$\begin{aligned}
\mathfrak{g}_0^{(1)} &= \mathbf{C}(E_{1,1} - E_{2,2}) \oplus \mathbf{C}(E_{2,2} - E_{3,3}) \oplus \mathbf{C}(E_{3,3} - E_{4,4}) \oplus \mathbf{C}E_{2,3} \oplus \mathbf{C}E_{3,2},\\
\mathfrak{g}_1^{(1)} &= \mathbf{C}E_{1,2} \oplus \mathbf{C}E_{1,3} \oplus \mathbf{C}E_{2,4} \oplus \mathbf{C}E_{3,4}, \quad \mathfrak{g}_2^{(1)} = \mathbf{C}E_{1,4},\\
\mathfrak{g}_0^{(2)} &= \mathbf{C}(E_{1,1} - E_{2,2}) \oplus \mathbf{C}(E_{2,2} - E_{3,3}) \oplus \mathbf{C}(E_{3,3} - E_{4,4})\\
&\qquad \oplus \mathbf{C}E_{1,2} \oplus \mathbf{C}E_{2,1} \oplus \mathbf{C}E_{3,4} \oplus E_{4,3},\\
\mathfrak{g}_2^{(2)} &= \mathbf{C}E_{1,3} \oplus \mathbf{C}E_{1,4} \oplus \mathbf{C}E_{2,3} \oplus \mathbf{C}E_{1,4}.
\end{aligned}$$

They satisfy Conditions ($\bigstar$).

Applying Proposition 2.3.1.1, one gets $\mathfrak{l}_1 = \mathbf{C}E_{1,3} \oplus \mathbf{C}E_{2,4}$ and

$$\mathfrak{n}_1 = \left\{ \begin{pmatrix} 0 & 0 & * & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\}, \quad \mathfrak{n}_2 = \left\{ \begin{pmatrix} 0 & 0 & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \right\}.$$

2.3.2. Order on the two nilpotent orbits

We recall that if f is nilpotent in $\mathfrak{g}$ and if $\mathbf{O}$ denotes the adjoint orbit of f, then $\mathbf{O}$ is *conical*, that is to say stable under the action of $\mathbf{G}_\mathrm{m}$ by scalar multiplication. When Conditions ($\bigstar$) hold, we can compare $\mathbf{O}_1$ and $\mathbf{O}_2$, the orbits of f_1 and f_2.

Lemma 2.3.2.1. *If Conditions ($\bigstar$) hold, then f_1 lies in $\mathfrak{g}_{-2}^{(2)}$ and it can be embedded in an $\mathfrak{sl}_2$-triple (e_1, h_1, f_1) such that $e_1 \in \mathfrak{g}_2^{(2)}$ and $h_1 \in \mathfrak{g}_0^{(2)}$.*

Proof. We have $[H_0, f_1] = 0$, hence $f_1 \in \mathfrak{g}_{-2}^{(2)}$. According to [EK05, Lemma 1.1], one can find an $\mathfrak{sl}_2$-triple (e_1', h_1', f_1) compatible with the H_1-grading. The decomposition $h_1' = h_1 + \sum_{j \neq 0} x_j$ holds, where h_1 is in $\mathfrak{g}_{0,0}$ and x_j is in $\mathfrak{g}_{0,j}$. In the same way, the decomposition $e_1' = e_1'' + \sum_{j \neq 2} y_j$ holds, where e_1'' is in $\mathfrak{g}_{2,2}$ and y_j is in $\mathfrak{g}_{2,j}$.

We have

$$-2f_1 = [h_1', f_1] = [h_1, f_1] + \sum_{j \neq 0} [x_j, f_1].$$

On the right-hand side, $[h_1, f_1]$ belongs to $\mathfrak{g}_{-2}^{(2)}$ and $[x_j, f_1]$ belongs to $\mathfrak{g}_{j-2}^{(2)}$ for $j \neq 0$. One can make the identification $[h_1, f_1] = -2f_1$ by comparing degrees, because f_1 is in $\mathfrak{g}^{(2)}_{-2}$. We can deduce the equality $h_1 = [e''_1, f_1]$ from the equality $h'_1 = [e'_1, f_1]$ by the same argument.

To conclude, we use an argument from [CM93, Proof of (3.3.10)]. Because h_1 commutes with H_1 and H_2, their three adjoint actions admit a common eigenbasis. We write the decomposition $e''_1 = e_1 + \sum_{j\neq 2} z_j$ as sum of eigenvectors for $\mathrm{ad}(h_1)$, where e_1 is in $\mathfrak{g}^{(1)}_2 \cap \mathfrak{g}^{(2)}_2 \cap \mathfrak{g}^{(h_1)}_2$ and z_j is in $\mathfrak{g}^{(1)}_2 \cap \mathfrak{g}^{(2)}_2 \cap \mathfrak{g}^{(h_1)}_j$. We have the equality

$$h_1 = [e''_1, f_1] = [e_1, f_1] + \sum_{j\neq 2}[z_j, f_1].$$

One sees that, for $j \neq 2$, $[z_j, f_1]$ belongs to $\mathfrak{g}^{(1)}_0 \cap \mathfrak{g}^{(2)}_0 \cap \mathfrak{g}^{(h_1)}_{j-2}$, and $[e_1, f_1]$ belongs to $\mathfrak{g}^{(1)}_0 \cap \mathfrak{g}^{(2)}_0 \cap \mathfrak{g}^{(h_1)}_0$. Since h_1 is in $\mathfrak{g}^{(1)}_0 \cap \mathfrak{g}^{(2)}_0 \cap \mathfrak{g}^{(h_1)}_0$, by identification one ets the equality $h_1 = [e_1, f_1]$.

We conclude that (e_1, h_1, f_1) is the desired $\mathfrak{sl}_2$-triple. □

Proposition 2.3.2.2. *Assume that Conditions* ($\bigstar$) *hold and fix an* $\mathfrak{sl}_2$*-triple as in Lemma 2.3.2.1, denoted by* (e_1, h_1, f_1)*. Then, the equality* $[f_0, e_1] = 0$ *holds and the orbit* $\mathbf{O}_{f_1}$ *is contained in the Zariski closure of* $\mathbf{O}_{f_2}$*.*

Proof. Because f_0 is in $\mathfrak{g}_{0,-2}$ and e_1 is in $\mathfrak{g}_{2,2}$, $[f_0, e_1]$ belongs to $\mathfrak{g}_{2,0}$, but this intersection is zero, so the bracket $[f_0, e_1]$ is zero.

The Lie algebra homomorphism $\mathfrak{sl}_2 \to \mathfrak{g}$ induced by the $\mathfrak{sl}_2$-triple exponentiates to a group homomorphism $\mathrm{SL}_2 \to G$. We get a cocharacter $\gamma : \mathbf{G}_\mathrm{m} \to G$ by restricting to the Cartan subgroup of SL_2 lifting the Cartan subalgebra $\mathbf{C}h$ of $\mathfrak{sl}_2$. The adjoint action composed with this cocharacter give a $\mathbf{G}_\mathrm{m}$-action on $\mathfrak{g}$ whose wheights correspond to the Dynkin grading of this $\mathfrak{sl}_2$-triple. In particular, $\mathrm{Ad}(\gamma(t))f = t^{-2}f$ for t in $\mathbf{G}_\mathrm{m}$.

Then the affine space $f_1 + \mathfrak{g}^{e_1}$ is equipped with the action

$$\rho(t)x \coloneqq t^2\,\mathrm{Ad}(\gamma(t))x \quad \text{for} \quad t \in \mathbf{G}_\mathrm{m} \quad \text{and} \quad x \in \mathfrak{g}.$$

Because $\mathfrak{g}^{e_1}$ is included in the nonegative part of the Dynkin grading according to Proposition 2.2.1.5, this action contracts to f_1, meaning that for any y in $\mathfrak{g}^{e_1}$, one has:

$$\rho(t)(f_1 + y) = f_1 + t\cdot y \longrightarrow f_1 \quad \text{as} \quad t \to 0.$$

Moreover, this actions preserves any nilpotent orbit. Because of the equality $[f_0, e_1] = 0$, f_2 is in $f_1 + \mathfrak{g}^{e_1}$, hence $f_1 = \lim\limits_{t\to 0}(\rho(t)f_2)$ lies in the closure of the orbit of f_2. □

2.3.3. Reduction by stages

If Conditions ($\bigstar$) hold, we get the nilpotent subalgebras $\mathfrak{n}_2$, $\mathfrak{n}_1$ and $\mathfrak{n}_0$ such that the corresponding unipotent subgroups of the reductive group G, denoted by N_2, N_1 and N_0, satisfy the semi-direct product decomposition

$$N_2 = N_1 \rtimes N_0.$$

For $i = 1, 2$, the group N_i acts on $\mathfrak{g}^*$ by the restriction of the coadjoint action and this actions has a moment map given by the restriction map

$$\pi_i : \mathfrak{g}^* \longrightarrow {\mathfrak{n}_i}^*, \quad \xi \longmapsto \xi|_{\mathfrak{n}_i}.$$

Denote by $\overline{\chi_i}$ the restriction of the linear form $\chi_i = (f_i|\bullet)$ to the subalgebra $\mathfrak{n}_i$ and denote by $\mathbb{O}_i \coloneqq \mathrm{Ad}^*(N_i)\overline{\chi_i}$ its coadjoint orbit.

By the proof of Proposition 2.3.1.1, the inclusion $\mathfrak{g}_{1,1} \subseteq \mathfrak{l}_i$ induces a symplectic isomorphism $\mathfrak{g}_{1,1} \cong \mathfrak{l}_i/{\mathfrak{l}_i}^{\perp,\omega_i}$. According to Proposition 2.2.2.2, we have the symplectic isomorphism

$$\sigma_i : \mathfrak{g}_{1,1} \longrightarrow \mathbb{O}_i, \quad v \longmapsto \overline{\chi_i} + \mathrm{ad}^*(v)\overline{\chi_i}.$$

Hence, the fibre of the orbit by the moment map is

$${\pi_i}^{-1}(\mathbb{O}_i^-) = -\chi_i + \mathrm{ad}^*(\mathfrak{g}_{1,1})\chi_i \oplus {\mathfrak{n}_i}^{\perp}.$$

We can equivalently consider the twisted moment map defined in (2.2.3.7):

$$\mu_i : \mathfrak{g} \times \mathfrak{g}_{1,1} \longrightarrow {\mathfrak{n}_i}^*, \quad (\xi, v) \longmapsto \pi_i(\xi) + \overline{\chi_i} + \mathrm{ad}^*(v)\overline{\chi_i}. \tag{2.3.3.1}$$

We have the following Poisson isomorphism of Hamiltonian reductions:

$${\mu_i}^{-1}(0)/\!/N_i \cong {\pi_i}^{-1}(\mathbb{O}_i^-)/\!/N_i.$$

Lemma 2.3.3.2. *The natural projection ${\mathfrak{n}_2}^* \twoheadrightarrow {\mathfrak{n}_1}^*$ induces a symplectic isomorphism $\mathbb{O}_2 \cong \mathbb{O}_1$ and the following triangle commutes:*

$$\begin{tikzcd}
 & {\mathfrak{n}_2}^* \arrow[dd, two heads] \\
\mathfrak{g}^* \times \mathfrak{g}_{1,1} \arrow[ur, "\mu_2"] \arrow[dr, "\mu_1"'] & \\
 & {\mathfrak{n}_1}^*.
\end{tikzcd}$$

The linear form χ_2 restricts to a character of $\mathfrak{n}_0$, denoted by $\overline{\chi}_0$.

Proof. The inclusion $\mathfrak{n}_1 \subseteq \mathfrak{n}_2$ implies the inclusion ${\mathfrak{n}_2}^{\perp} \subseteq {\mathfrak{n}_1}^{\perp}$. The decomposition $f_2 = f_1 + f_0$ where f_0 belongs to $\mathfrak{g}_{0,-2}$ and is orthogonal to $\mathfrak{n}_1$ with respect to $(\bullet|\bullet)$. Hence, we have the decomposition $\chi_2 = \chi_1 + \chi_0$ where χ_0 is in ${\mathfrak{n}_1}^{\perp}$ and $\overline{\chi}_2|_{\mathfrak{n}_1} = \overline{\chi}_1$, so the diagram is commutative.

The linear form $\overline{\chi_0} = \chi_2|_{\mathfrak{n}_0}$ is equal the inner product by f_0, that belongs to $\mathfrak{g}_{0,-2}$. Because $\mathfrak{n}_0 = \mathfrak{a} \oplus (\mathfrak{g}_0^{(1)} \cap \mathfrak{g}_{\geqslant 2}^{(1)})$ with $\mathfrak{a} \subseteq \mathfrak{g}_{0,1}$, we only need to check that

$$\overline{\chi}_0([\mathfrak{a}, \mathfrak{a}]) = \chi_2([\mathfrak{a}, \mathfrak{a}]) = 0.$$

This is true because $\mathfrak{a}$ is contained in the isotropic subspace $\mathfrak{l}_2$. $\square$

The following theorem is a new formulation of [GJ24, Main Theorems 1 and 2] under conditions ($\bigstar$). Note that $\mathfrak{l}_1$ and $\mathfrak{l}_2$ may not be Lagrangian, in contrast to [GJ24].

Theorem 2.3.3.3. *Assume Conditions ($\bigstar$). Then Proposition 2.3.1.1 holds and we can use the objects introduced just above. The following properties hold:*

1. *the action of N_0 descends to the quotient ${\mu_1}^{-1}(0)/\!/N_1$ and there is an induced moment map*
$$\mu_0 : {\mu_1}^{-1}(0)/\!/N_1 \longrightarrow {\mathfrak{n}_0}^*, \quad [\xi, v] \longmapsto \xi|_{\mathfrak{n}_0} + \overline{\chi_0},$$
where $[\xi, v]$ denotes the N_1-orbit of (ξ, v) in ${\mu_1}^{-1}(0)$,

2. *the action induces an N_0-equivariant isomorphism $N_0 \times S_2 \cong {\mu_0}^{-1}(0)$, where the left-hand side is equipped with the action of left multiplication on N_0,*

3. *there is a Poisson isomorphism*
$${\mu_2}^{-1}(0)/\!/N_2 \cong {\mu_0}^{-1}(0)/\!/N_0$$
induced by the inclusion ${\mu_2}^{-1}(0) \subseteq {\mu_1}^{-1}(0)$.

We will prove Theorem 2.3.3.3 and Corollary 2.1.4.7 by using Proposition 2.1.4.4. On the coordinate ring side, we getthe following statement.

Corollary 2.3.3.4. *The co-homomorphism*

$$\begin{aligned} (\mathscr{P}(\mathfrak{g}, f_1)/I_0)^{N_0} &\longrightarrow \mathscr{P}(\mathfrak{g}, f_2) \\ (F \bmod I_1) \bmod I_0 &\longmapsto F \bmod I_2 \end{aligned}$$

is a Poisson algebra isomorphism, where I_0 the ideal of $\mathscr{P}(\mathfrak{g}, f_1)$ spanned by

$$x \otimes 1 + \chi_2(x)1 \bmod I_1 \quad \textit{for} \quad x \in \mathfrak{n}_0.$$

We introduce some tools to prove Theorem 2.3.3.3. For $i = 1, 2$, embed the nilpotent element f_i in an $\mathfrak{sl}_2$-triple (e_i, h_i, f_i) such that e_i belongs to $\mathfrak{g}_2^{(i)}$ and h_i belongs to $\mathfrak{g}_0^{(i)}$, and let $S_i \coloneqq -\chi_i + [\mathfrak{g}, e_i]^\perp$ be the corresponding Slodowy slice. Moreover assume that the triple (e_1, h_1, f_1) is provided by Lemma 2.3.2.1. According to Theorem 2.2.3.3, the map

$$\alpha_i : N_i \times S_i \longrightarrow {\pi_i}^{-1}(\mathbb{O}_i^-), \quad (g, \xi) \longmapsto \mathrm{Ad}^*(g)\xi$$

is an algebraic isomorphism.

Lemma 2.3.3.5. *The isomorphism $\alpha_1 : N_1 \times S_1 \to {\mu_1}^{-1}(0)$ restricts to an isomorphism*

$$N_1 \times (-\chi_2 + [\mathfrak{g}, e_1]^\perp \cap {\mathfrak{n}_0}^\perp) \cong {\pi_2}^{-1}(\mathbb{O}_2^-).$$

Proof. We first check the inclusion

$$\phi_1(N_1 \times (-\chi_2 + [\mathfrak{g}, e_1]^\perp \cap \mathfrak{n}_0{}^\perp)) \subseteq \pi_2{}^{-1}(\mathbb{O}_2^-).$$

Since $\pi_2{}^{-1}(\mathbb{O}_2^-)$ is N_1-stable, it is enough to check that $-\chi_2 + [\mathfrak{g}, e_1]^\perp \cap \mathfrak{n}_0{}^\perp$ is contained in $\pi_2{}^{-1}(\mathbb{O}_2^-) = -\chi_2 + \mathrm{ad}^*(\mathfrak{g}_{1,1})\chi_2 \oplus \mathfrak{n}_2{}^\perp$. It is clear because $[\mathfrak{g}, e_1]^\perp$ is in $\mathfrak{n}_1{}^\perp$ and $\mathfrak{n}_1{}^\perp \cap \mathfrak{n}_0{}^\perp = \mathfrak{n}_2{}^\perp$.

We have an inclusion of two smooth irreducible closed varieties, if they have the same dimension then they are equal and by restriction we have the desired isomorphism. The subalgebra $\mathfrak{n}_0$ is contained in $\mathfrak{g}^{f_1}$ that does not intersect $[\mathfrak{g}, e_1]$ according Proposition 2.2.1.2, hence

$$\dim(N_1 \times (-\chi_2 + [\mathfrak{g}, e_1]^\perp \cap \mathfrak{n}_0{}^\perp)) = \dim \mathfrak{n}_1 + \dim \mathfrak{g} - \dim[\mathfrak{g}, e_1] - \dim \mathfrak{n}_0.$$

Because α_1 is an isomorphism, we get

$$\dim \mathfrak{n}_1 - \dim[\mathfrak{g}, e_1] = \dim \mathfrak{g}_{1,1} - \dim \mathfrak{n}_1,$$

hence

$$\begin{aligned} \dim(N_1 \times (-\chi_2 + [\mathfrak{g}, e_1]^\perp \cap \mathfrak{n}_0{}^\perp)) &= \dim \mathfrak{g}_{1,1} + \dim \mathfrak{g} - \dim \mathfrak{n}_1 - \dim \mathfrak{n}_0 \\ &= \dim \pi_2{}^{-1}(\mathbb{O}_2^-). \end{aligned}$$

This equality concludes the proof. $\square$

We can use this lemma to prove the main theorem.

Proof of Theorem 2.3.3.3. Because of the compatibility of the moment maps given by Lemma 2.3.3.2, one can apply Lemma 2.1.4.3 from the general theory of Hamiltonian reduction by stages. It proves Item 1 of the theorem.

Set

$$\widetilde{\mu_0} : S_1 \longrightarrow \mathfrak{n}_0{}^*, \quad \xi \longmapsto \xi|_{\mathfrak{n}_0} + \overline{\chi_0}.$$

Clearly the following triangle commutes:

$$\begin{tikzcd} S_1 \arrow[d, "\wr"'] \arrow[dr, "\widetilde{\mu_0}"] & \\ \mu_1{}^{-1}(0)/\!/N_1 \arrow[r, "\mu_0"'] & \mathfrak{n}_0{}^* \end{tikzcd}$$

The fibre of 0 is:

$$\widetilde{\mu_0}^{-1}(0) = -\chi_2 + [\mathfrak{g}, e_1]^\perp \cap \mathfrak{n}_0{}^\perp.$$

Lemma 2.3.3.5 gives an isomorphism

$$N_1 \times \widetilde{\mu_0}^{-1}(0) \cong \mu_2{}^{-1}(0),$$

hence the natural map ${\mu_2}^{-1}(0)//N_1 \to {\mu_0}^{-1}(0)$ is an isomorphism. By Proposition 2.1.4.4, Item 3 of the theorem is proved.

The isomorphism

$$\begin{aligned} N_1 \times N_0 \times S_2 &\longrightarrow {\pi_2}^{-1}(\mathbb{O}_2^-), \\ (g_1, g_0, \xi) &\longmapsto \mathrm{Ad}^*(g_1 g_0)\xi \end{aligned}$$

induces an isomorphism

$$N_0 \times S_2 \cong {\pi_2}^{-1}(\mathbb{O}_2^-)//N_1 \cong {\mu_0}^{-1}(0)$$

after taking quotient by N_1, that is to say Item 2 of the theorem. □

2.3.4. Functorial reduction by stages

Let X be an affine Poisson scheme equipped with an action of the group G by Poisson automorphisms and a moment map $\phi_X : X \to \mathfrak{g}^*$. There is an action of N_2 on X and a diagonal action on $X \times \mathfrak{g}_{1,1}$ with a moment map

$$\mu_{X,2} : X \times \mathfrak{g}_{1,1} \longrightarrow {\mathfrak{n}_2}^*, \quad (x, v) \longmapsto \pi_2(\phi_X(x)) + \overline{\chi}_2 + \mathrm{ad}^*(v)\overline{\chi}_2.$$

There is also an action of N_1 and a moment map

$$\mu_{X,1} : X \times \mathfrak{g}_{1,1} \longrightarrow {\mathfrak{n}_1}^*, \quad (x, v) \longmapsto \pi_1(\phi_X(x)) + \overline{\chi}_1 + \mathrm{ad}^*(v)\overline{\chi}_1$$

such that the following diagram is commutative:

$$\begin{tikzcd} & {\mathfrak{n}_2}^* \arrow[dd, two heads] \\ X \times \mathfrak{g}_{1,1} \arrow[ur, "\mu_{X,2}"] \arrow[dr, "\mu_{X,1}"'] & \\ & {\mathfrak{n}_1}^*. \end{tikzcd}$$

Corollary 2.3.4.1. *Assume Conditions (★). The following properties hold:*

1. *the action of N_0 descends to the quotient ${\mu_{X,1}}^{-1}(0)//N_1$ and there is an induced moment map $\mu_{X,0} : {\mu_{X,1}}^{-1}(0)//N_1 \to {\mathfrak{n}_0}^*$,*

2. *the action induces an N_0-equivariant isomorphism*

$$N_0 \times {\phi_X}^{-1}(S_2) \cong {\mu_{X,0}}^{-1}(0),$$

where the left-hand side is equipped with the action of left multiplication on N_0,

3. *there is a Poisson isomorphism*

$${\mu_{X,2}}^{-1}(0)//N_2 \cong {\mu_{X,0}}^{-1}(0)//N_0$$

induced by the inclusion ${\mu_{X,2}}^{-1}(0) \subseteq {\mu_{X,1}}^{-1}(0)$.

Proof. The argument is similar to the one of Theorem 2.3.3.3. The general theory of reduction by stages (Lemma 2.1.4.3) gives the action of N_0 and a moment map. The isomorphism isomorphism

$$N_1 \times (-\chi_2 + [\mathfrak{g}, e_1]^\perp \cap \mathfrak{n}_0{}^\perp) \cong \pi_2{}^{-1}(\mathcal{O}_2^-),$$

thanks to Lemma 2.2.4.2, implies an isomorphism

$$N_1 \times \big((-\chi_2 + [\mathfrak{g}, e_1]^\perp \cap \mathfrak{n}_0{}^\perp) \times_{\mathfrak{g}^*} X\big) \cong \mu_{X,2}{}^{-1}(0).$$

The two other statements follow as for Theorem 2.3.3.3. $\square$

3 - Examples of reduction by stages

In this section, we provide several examples in classical type for the Hamiltonian reduction by stage. For the majority of them, these examples are from [GJ24]. Proposition 3.1.2.2 is from [GJ25].

It is well-known that in all classical types, the nilpotent orbits of a simple Lie algebra are parametrised by partitions [CM93, Chapter 5]. A partition of the positive integer n is a nonincreasing sequences of positive integers

$$a_\bullet := (a_1, \dots, a_r) \quad \text{such that} \quad \sum_{i=1}^{r} a_i = n.$$

When in a subsequence $(a_i, \dots, a_j)$ of $a_\bullet$, all a_k's are equal to an integer a, the subsequence is compactly written as (a^{j-i+1}). A partition is represented by its associated *Young diagram*.

To classify good gradings in classical types, Elashvili and Kac introduced the notion of *pyramid* in [EK05]. Roughly speaking, these pyramids are built by shifting the rows in a Young diagram and fixing a bijective labeling of the boxes by numbers between 1 and n. To draw our pyramids, we use the same conventions as in [BG07]. We draw a pyramid in the xy-plane so that the rows are parallel to the x-axis. We assume that the box size is twice the unit length in this coordinate system. From this pyramid, one can build a nilpotent element f and a semisimple element H defining a good grading for f.

Not that we follow the convention of [AvEM24], that is slightly different from [BG07]. In [BG07], they fixe e and they deal with good gradings such that $e \in \mathfrak{g}_2$. In [AvEM24] and the present thesis, one takes an alternative convention by fixing f such that f belongs to $\mathfrak{g}_{-2}$.

3.1. Examples in type A

Let $r \geqslant 1$ be an integer, set $n := r+1$ and let $\mathfrak{sl}_n$ be the simple Lie algebra of type A_r.

3.1.1. Hook-type partitions

We consider a hook-type partition of n:

$$a_\bullet^{(\ell)} := (\ell, 1^{n-\ell}), \quad 1 \leqslant \ell \leqslant n.$$

The following left-aligned pyramid of shape $a_\bullet^{(\ell)}$ represented in Figure 3.1 defines a nilpotent element

$$f^{(\ell)} := E_{2,1} + E_{3,2} + \dots + E_{\ell,\ell-1}.$$

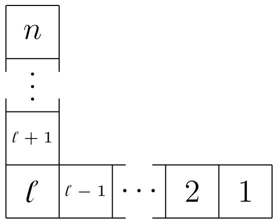


Figure 3.1: Pyramid for hook partition $(\ell, 1, \ldots, 1)$

and an even good grading corresponding to the coadjoint action of

$$H^{(\ell)} := \operatorname{diag}\big(-\frac{\ell-1}{2}, \ldots, -\frac{\ell-1}{2}, -\frac{\ell-1}{2}+2, \ldots, \frac{\ell-1}{2}-2, \frac{\ell-1}{2}\big).$$

For any integers $1 \leqslant \ell_1 < \ell_2 \leqslant n$, the corresponding two good gradings satisfy Conditions ($\bigstar$).

The positive part is given by:

$$\mathfrak{n}^{(\ell)} := \operatorname{Span}\{E_{i,j} \mid 1 \leqslant i \leqslant \ell-1,\ i < j \leqslant n\}.$$

Proposition 3.1.1.1. *For any integers $1 \leqslant \ell_1 < \ell_2 \leqslant n$, consider the hook-type nilpotent elements*

$$f_1 := f^{(\ell_1)} \quad \textit{and} \quad f_2 := f^{(\ell_2)}.$$

Reduction by stages holds between their Slodowy slices.

3.1.2. Hook on top

Consider two partitions of n the form

$$a_\bullet^{(1)} := (a_1, \ldots, a_r, a, 1^b) \quad \text{and} \quad a_\bullet^{(2)} := (a_1, \ldots, a_r, a+1, 1^{b-1}).$$

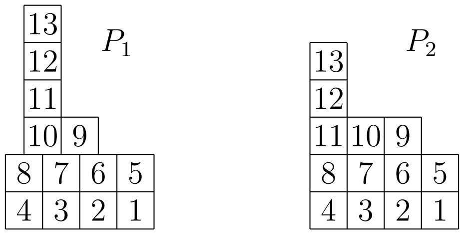
These partitions start with the same element $a_1, \ldots, a_r$ and finish with two different hook-type partitions, $(a, 1^b)$ and $(a+1, 1^{b-1})$.

We associate to theses partitions the pyramids P_1 and P_2 as follows. Both pyramids start with the same first r rows of length $(a_1, \ldots, a_r)$, that we draw left-aligned and with the same labels. The hook-type part $(a, 1^b)$ of P_1 is also left aligned, but there is a half-box shift between the rows $(a_1, \ldots, a_r)$ and the hook-type part. Fix an arbitrary labeling of these boxes. The hook-type part of P_2 is built in two steps:

1. shift the hook-type part $(a, 1^b)$ of P_1 by half a box to the right,

2. move the boxes corresponding to the part (1^b) by one box to the left and one box down.

Figure 3.2: Examples of pyramids



Example 3.1.2.1. Consider the following partitions of $n = 9$:

$$a_\bullet^{(1)} = (4, 4, 2, 1^3) \quad \text{and} \quad a_\bullet^{(2)} = (4, 4, 3, 1^2).$$

The associated pyramids are drawn in Figure 3.2.

Proposition 3.1.2.2. *For $i = 1, 2$, let f_i be a nilpotent matrix and H_i be the diagonal element in $\mathfrak{sl}_n$ corresponding to the pyramid P_i described above. Then, the pairs (f_1, H_1) and (f_2, H_2) satisfy the condition ($\bigstar$) and reduction by stages holds.*

Proof. From the left to the right, and from the bottom to the top, denote by $j_1, \dots, j_a, j_{a+1}, \dots, j_{a+b}$ the labels in the hook-type part (in Example 3.1.2.1, these labels are $9, 10, 11, 12, 13$).

Take $1 \leqslant j, k \leqslant n$. Let $x_j^{(i)}$, respectively $x_k^{(i)}$, be the abscissa of the center of the box labeled with j, respectively with k. Then

$$[H_i, E_{j,k}] = (x_j^{(i)} - x_k^{(i)})E_{j,k}.$$

The nilpotent element f_i is the sum of elementary matrices $E_{j,k}$ such that the boxes labeled by j and k are in the same row and $x_j^{(i)} - x_k^{(i)} = -2$. Then $f_0 = f_2 - f_1$ is equal to E_{j_{a+1}, j_a} and since $x_{j_{a+1}}^{(i)} = x_{j_a}^{(i)}$, one has $[H_1, f_0] = 0$.

Let us prove that for any labels j, k,

$$\begin{aligned}
&\text{if} \quad x_j^{(1)} - x_k^{(1)} \geqslant 2 \quad \text{then} \quad x_j^{(2)} - x_k^{(2)} \geqslant 1, \\
&\text{if} \quad x_j^{(2)} - x_k^{(2)} \geqslant 1 \quad \text{then} \quad x_j^{(1)} - x_k^{(1)} \geqslant 0, \\
&\text{if} \quad |x_j^{(1)} - x_k^{(1)}| = 1 \quad \text{then} \quad |x_j^{(1)} - x_k^{(1)}| \in \{0, 1, 2\}.
\end{aligned}$$

If j labels one box in the hook-type part and k labels a box in the first r rows, it is because the relative position between a box in the hook-type part and a box in the first r rows only vary of half a box when P_1 is turned into P_2. The other cases are clear.

Hence the conditions ($\bigstar$) are all satisfied. $\square$

Remark 3.1.2.3. This family of examples contains cases which are already known, see Table 1.2. However, some reduction by stages from [FFFN24, FKN24] are not covered by this family. For example, $\mathfrak{g} = \mathfrak{sl}_4$, f_1 being a rectangular nilpotent element (partition $(2,2)$) and f_2 being a subregular one.

3.2. Examples in other types

3.2.1. Type B

Set $n \coloneqq 2r+1$ for $r \geqslant 2$ an integer. We realize the simple Lie algebra of type B_r as the following subalgebra of $\mathfrak{sl}_n$:

$$\mathfrak{so}_n \coloneqq \{x \in \mathfrak{sl}_n \mid x^{\mathsf{T}} K + Kx = 0\}, \quad K \coloneqq \begin{pmatrix} {\scriptstyle (0)} & & {\scriptstyle 1} \\ & \cdot^{\cdot^{\cdot}} & \\ {\scriptstyle 1} & & {\scriptstyle (0)} \end{pmatrix}.$$

where x^{T} denotes the transpose of x. As in [EK05, BG07], we take the symmetric set

$$I_n \coloneqq \{-r, \dots, -2, -1, 0, 1, 2, \dots, r\}$$

as indexation for the canonical basis of $\mathbf{C}^n$, with the following order:

$$v_{-r} = (1, 0, \dots, 0), \quad v_{-r+1} = (0, 1, \dots, 0), \quad \dots \quad v_r = (0, 0, \dots, 1).$$

We change the numbering of the elementary matrices $E_{i,j}$ to have i, j in I_n and to respect the order we have chosen for the basis.

Consider the regular nilpotent element

$$f_2 \coloneqq \sum_{i=0}^{r-1} E_{i+1,i} - \sum_{i=0}^{r-1} E_{-i,-i-1}$$

corresponding to the regular orthogonal partition $a_{\bullet}^{(2)} \coloneqq (n)$. We consider the Dynkin grading which is induced by the adjoint action of the diagonal matrix

$$H_2 \coloneqq \operatorname{diag}(2r, \dots, 4, 2, 0, -2, -4, \dots, -2r).$$

The positive part is given by:

$$\mathfrak{n}_2 = \operatorname{Span}\{E_{i,j} \mid i < j \in I_n\} \cap \mathfrak{so}_n.$$

We consider the subregular orthogonal partition $a_{\bullet}^{(1)} \coloneqq (n-2, 1^2)$ of n. We associate to it the orthogonal pyramid of Figure 3.3. From this pyramid we can construct the nilpotent element

$$f_1 \coloneqq \sum_{i=0}^{r-2} E_{i+1,i} - \sum_{i=0}^{r-2} E_{-i,-i-1}$$

of type $a_\bullet^{(1)}$ and an even good grading for f_1 induced by the adjoint action of

$H_1 := \mathrm{diag}(2r-2, 2r-2, 2r-4, 2r-6, \dots, 2, 0, 2, \dots, 6-2r, 4-2r, 2-2r, 2-2r)$.

The positive part is given by:

$$\mathfrak{n}_1 = \mathrm{Span}\,\{E_{i,j} \mid i < j \in I_n,\ i < r-1,\ j > -r+1\} \cap \mathfrak{so}_n.$$

Conditions ($\bigstar$) are satisfied by these two good gradings.

Proposition 3.2.1.1. *The Slodowy slice to a regular nilpotent element in $\mathfrak{so}_n$ is a Hamiltonian reduction of the slice to a subregular nilpotent element in $\mathfrak{so}_n$.*

3.2.2. Type C

Set $n = 2r$ for $r \geqslant 3$ an integer. We realize the simple Lie algebra of type C_r as the following subalgebra of $\mathfrak{sl}_n$:

$$\mathfrak{sp}_n := \{x \in \mathfrak{sl}_n \mid x^{\mathsf{T}} J + Jx = 0\}, \quad J := \begin{pmatrix} & & & & 1 \\ & (0) & & \iddots & \\ & & -1 & 1 & \\ & \iddots & & & (0) \\ -1 & & & & \end{pmatrix}.$$

As in [EK05, BG07], we take the symmetric set

$$I_n := \{-r, \dots, -2, -1, 1, 2, \dots, r\}$$

as indexation for the canonical basis of $\mathbf{C}^n$, with the following order:

$$v_{-r} = (1, 0, \dots, 0), \quad v_{-r+1} = (0, 1, \dots, 0), \quad \dots \quad v_r = (0, 0, \dots, 1).$$

We change the numbering of the elementary matrices $E_{i,j}$ to have i, j in I_n and to respect the order we have chosen for the basis.

Consider the regular nilpotent element

$$f_2 := E_{1,-1} + \sum_{i=1}^{r-1} E_{i+1,i} - \sum_{i=1}^{r-1} E_{-i,-i-1}$$

corresponding to the partition $a_\bullet^{(2)} := (n)$. We consider the Dynkin grading induced by the adjoint action of

$$H_2 = h_2 =:= \mathrm{diag}(2r-1, \dots, 3, 1, -1, -3, \dots, 1-2r).$$

It is even and the corresponding positive part is given by:

$$\mathfrak{n}_2 = \mathrm{Span}\,\{E_{i,j} \mid i < j \in I_n\} \cap \mathfrak{sp}_n.$$

We consider the symplectic partition $a_\bullet^{(1)} := (2^2, 1^{n-4})$ of n. It corresponds to the second smallest nonzero nilpotent orbit, after the minimal one. We

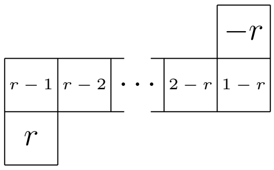


Figure 3.3: Orthogonal pyramid for subregular parition $(n-2,1,1)$

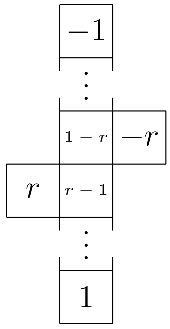


Figure 3.4: Symplectic pyramid for partition $(2,2,1,\dots,1)$

associate to it the symplectic pyramid of Figure 3.4. From this pyramid we can construct the nilpotent element

$$f_1 \coloneqq E_{r,r-1} - E_{-r+1,-r}$$

of type $a_{\bullet}^{(1)}$ and an even good grading for f_1 induced by the adjoint action of the diagonal matrix $H_1 \coloneqq \operatorname{diag}(2,0,\dots,0,0,\dots,0,-2)$. The corresponding positive part is:

$$\mathfrak{n}_1 = \operatorname{Span}\{E_{-r,i},\ E_{j,r} \mid i > -r,\ j < r\} \cap \mathfrak{sp}_n.$$

The two good gradings satisfy Conditions ($\bigstar$).

Proposition 3.2.2.1. *In type* C_r*,* $r \geqslant 3$*, the regular Slodowy slice is a Hamiltonian reduction of the slice corresponding to* $(2^2, 1^{n-4})$*.*

3.2.3. Type G

Let $\mathfrak{g}$ be the simple Lie algebra of type G_2. For a concrete construction of this exceptionnal Lie algebra, one can see [Hum72, Section 19.3], where $\mathfrak{g}$ is build as a subalgebra of the classical simple algebra of type B_3. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and Δ_+ be a set of positive roots of $\mathfrak{g}$ with simple roots $\Pi = \{\alpha_1, \alpha_2\}$, where α_1 is a short root and α_2 is a long root. Then

$$\Delta_+ = \{\alpha_1, \alpha_2, \alpha_1+\alpha_2, 2\alpha_1+\alpha_2, 3\alpha_1+\alpha_2, 3\alpha_1+2\alpha_2\}.$$

Choose positive root vectors e_α, for $\alpha \in \Delta_+$, and negative root vectors f_α, for $\alpha \in \Delta_- = -\Delta_+$. Using the normalized symmetric invariant bilinear form on $\mathfrak{g}$, we have an isomorphism $\mathfrak{h} \cong \mathfrak{h}^*$. Denote by h_i the vector in $\mathfrak{h}$ corresponding to α_i so that $\alpha_i(h_j) = (\alpha_i|\alpha_j)$. Let

$$f_1 \coloneqq f_{2\alpha_1+\alpha_2}, \quad f_2 \coloneqq f_{\alpha_2} + f_{2\alpha_1+\alpha_2}.$$

Then f_1 is a nilpotent element labeled by $\widetilde{\mathrm{A}}_1$ in the Bala-Carter theory and f_2 is a subregular nilpotent element. We have $\mathfrak{sl}_2$-triples (e_k, x_k, f_k) in $\mathfrak{g}$, for $k = 1, 2$, where

$$x_1 \coloneqq 6h_1 + 3h_2, \quad x_2 \coloneqq 6h_1 + 4h_2.$$

Equivalently, $x_1 = \varpi_1^\vee$ and $x_2 = 2\varpi_2^\vee$, where $\varpi_i^\vee$ is the i-the fundamental coweight in $\mathfrak{h}$ defined by $\alpha_i(\varpi_j^\vee) = \delta_{i,j}$. Let $\mathfrak{g} = \bigoplus_{j \in \mathbf{Z}} \mathfrak{g}_j^{(k)}$ be the Dynkin gradings on $\mathfrak{g}$ defined by $\mathrm{ad}(x_k)$ for $k = 1, 2$. Then

$$\begin{aligned} \mathfrak{g} &= \mathfrak{g}_{-3}^{(1)} \oplus \mathfrak{g}_{-2}^{(1)} \oplus \mathfrak{g}_{-1}^{(1)} \oplus \mathfrak{g}_0^{(1)} \oplus \mathfrak{g}_1^{(1)} \oplus \mathfrak{g}_2^{(1)} \oplus \mathfrak{g}_3^{(1)} \\ &= \mathfrak{g}_{-4}^{(2)} \oplus \mathfrak{g}_{-2}^{(2)} \oplus \mathfrak{g}_0^{(2)} \oplus \mathfrak{g}_2^{(2)} \oplus \mathfrak{g}_4^{(2)} \end{aligned}$$

and

$$\begin{aligned} &\mathfrak{g}_1^{(1)} = \mathbf{C}e_{\alpha_1} \oplus \mathbf{C}e_{\alpha_1+\alpha_2}, \quad \mathfrak{g}_2^{(1)} = \mathbf{C}e_{2\alpha_1+\alpha_2}, \quad \mathfrak{g}_3^{(1)} = \mathbf{C}e_{3\alpha_1+\alpha_2} \oplus \mathbf{C}e_{3\alpha_1+2\alpha_2}, \\ &\mathfrak{g}_2^{(2)} = \mathbf{C}e_{\alpha_2} \oplus \mathbf{C}e_{\alpha_1+\alpha_2} \oplus \mathbf{C}e_{2\alpha_1+\alpha_2} \oplus \mathbf{C}e_{3\alpha_1+\alpha_2}, \quad \mathfrak{g}_4^{(2)} = \mathbf{C}e_{3\alpha_1+2\alpha_2}. \end{aligned}$$

These good gradings satisfy Conditions ($\bigstar$). We may take $\mathfrak{l}_1 = \mathbf{C}e_{\alpha_1+\alpha_2}$ as a Lagrangian subspace in $\mathfrak{g}_1^{(1)}$ with respect to the symplectic form $(x, y) \mapsto (f_1|[x, y])$. Set

$$\mathfrak{n}_1 = \mathfrak{l}_1 \oplus \mathfrak{g}_2^{(1)} \oplus \mathfrak{g}_3^{(1)}, \quad \mathfrak{n}_2 = \mathfrak{g}_2^{(2)} \oplus \mathfrak{g}_4^{(2)}.$$

Proposition 3.2.3.1. *Let $\mathfrak{g}$ be the simple Lie algebra of type G_2. Then the Slodowy slice to a subregular nilpotent element in $\mathfrak{g}$ is a Hamiltonian reduction of the slice to a nilpotent element in $\mathfrak{g}$ whose Bala-Carter label is $\widetilde{\mathrm{A}}_1$.*

4 - Quantum reduction by stages and finite W-algebras

4.1. Reduction by stages for noncommutative algebras

In this section we recall standard definitions and facts about quantisations of affine Poisson schemes by filtered associative algebras, for instance see [ACET24, Section 2]. We develop a general theory of $\mathbf{G}_{\mathrm{m}}$-equivariant actions of a graded algebraic group on a filtered vector space and a sufficient condition to compute the invariants of the graded module associated with a filtered module on a graded unipotent group (Proposition 4.1.2.5). We recall standard definitions and facts about quantum Hamiltonian reductions [GG06, Section 7], but in a slightly more general setting for which a pair of groups is considered, and we give a sufficient condition for the quantum Hamiltonian reduction to be the quantisation of a geometric Hamiltonian reduction, when the acting group is unipotent (Theorem 4.1.3.8). We introduce Hamiltonian reduction by stages in the noncommutative setting.

4.1.1. Quantisation of a Poisson scheme

Let $V_\bullet$ be a *filtered vector space*, that is to say a vector space V equipped with an increasing filtration by linear subspaces V_p, for p in $\mathbf{Z}$. The associated graded vector space is

$$\operatorname{gr}_\bullet V = \bigoplus_{p \in \mathbf{Z}} \operatorname{gr}_p V, \quad \text{where} \quad \operatorname{gr}_p V \coloneqq V_p / V_{p-1}.$$

For v in V_p, denote by $[v]_p$ its class in the quotient $\operatorname{gr}_p V$.

Let $\mathcal{A}_\bullet$ be a *filtered algebra*, that is to say a (possibly noncommutative) associative unital algebra $\mathcal{A}$ equipped with an increasing filtration such that for any p, q in $\mathbf{Z}$, there is the inclusion $\mathcal{A}_p \cdot \mathcal{A}_q \subseteq \mathcal{A}_{p+q}$. Then $\operatorname{gr}_\bullet \mathcal{A}$ is a graded algebra. The filtered algebra $\mathcal{A}_\bullet$ is called *almost commutative of degree* $-k$, for $k \geqslant 1$ an integer, if for any p, q in $\mathbf{Z}$, there is the inclusion

$$[\mathcal{A}_p, \mathcal{A}_q] \subseteq \mathcal{A}_{p+q-k}.$$

For such an almost commutative algebra, $\operatorname{gr}_\bullet \mathcal{A}$ is commutative and equipped with the following Poisson bracket:

$$\{[a]_p, [b]_q\} \coloneqq [ab - ba]_{p+q-k},$$

where a is in $\mathcal{A}_p$ and b is in $\mathcal{A}_q$.

Let X be an affine Poisson scheme such that its coordinate ring is equipped with an algebra $\mathbf{Z}$-grading:

$$\mathbf{C}[X]_\bullet = \bigoplus_{p \in \mathbf{Z}} \mathbf{C}[X]_p.$$

We assume that the Poisson bracket is homogeneous of degree $-k$: for any p, q in $\mathbf{Z}$, there is the inclusion

$$\{\mathbf{C}[X]_p, \mathbf{C}[X]_q\} \subseteq \mathbf{C}[X]_{p+q-k}.$$

A *quantisation* of X is the data of a filtered algebra $\mathscr{A}_\bullet$ that is almost commutative of degree $-k$ and a Poisson graded isomorphism $\mathbf{C}[X]_\bullet \cong \operatorname{gr}_\bullet \mathscr{A}$.

Example 4.1.1.1. Let $\mathfrak{g}$ be a finite dimensional Lie algebra. The *universal enveloping algebra* $\mathcal{U}(\mathfrak{g})$ is the quotient of the tensor algebra $\bigoplus_{n\geqslant 0} \mathfrak{g}^{\otimes n}$ by the two-sided ideal spanned by $x \otimes y - y \otimes x - [x, y]$ for x, y in $\mathfrak{g}$.

The grading on the tensor algebra induces a filtration on $\mathcal{U}(\mathfrak{g})$, called the *PBW (Poincaré–Birkhoff–Witt) filtration*. It is in fact almost commutative of degree -1 and it is a quantisation of the Poisson variety $\mathfrak{g}^* = \operatorname{Spec}(\operatorname{Sym} \mathfrak{g})$ defined in Example 2.1.2.1.

Example 4.1.1.2. Let (V, ω) be a symplectic vector space. The associated *Weyl algebra* $\mathcal{W}(V)$ is the quotient of the tensor algebra $\bigoplus_{n\geqslant 0} V^{\otimes n}$ by the two-sided ideal spanned by $v \otimes w - w \otimes v - \omega(v, w)1$. It is a quantisation of the Poisson variety $V \cong \operatorname{Spec}(\operatorname{Sym} V)$ defined in Example 2.1.2.3.

4.1.2. Graded group, filtered module and invariants

Recall that the data of a $\mathbf{Z}$-grading $V_\bullet$ on a vector space V is equivalent to the data of $\mathbf{G}_\mathrm{m}$-module structure on V. If $\rho : V \to \mathbf{C}[t, t^{-1}] \otimes_\mathbf{C} V$ is the comodule structure, then

$$V_p = \{v \in V \mid \rho(v) = t^p \otimes v\} \qquad \text{for} \quad p \in \mathbf{Z}.$$

If X is an affine scheme equipped with an action $\rho : \mathbf{G}_\mathrm{m} \times X \to X$, then $\mathbf{C}[X]$ is graded by

$$\mathbf{C}[X]_p = \{F \in \mathbf{C}[X] \mid \rho^*(F) = t^p \otimes F\} \qquad \text{for} \quad p \in \mathbf{Z}.$$

Let N be an algebraic group and $\mathfrak{n}$ be its Lie algebra. The data of a Lie algebra $\mathbf{Z}$-grading on $\mathfrak{n}$ is equivalent to the data of an algebraic action of $\mathbf{G}_\mathrm{m}$ on $\mathfrak{n}$ by Lie algebra automorphisms. By analogy, a grading on N is the data of an action of $\mathbf{G}_\mathrm{m}$ on N by algebraic group automorphisms. Note that a grading on N induces a grading on $\mathfrak{n}$.

If N is graded by a $\mathbf{G}_\mathrm{m}$-action, denote by N^- its *opposite graded group*, that is defined as the same underlying algebraic group equipped the inverse $\mathbf{G}_\mathrm{m}$-action.

Example 4.1.2.1. Let G be a connected algebraic group whose Lie algebra $\mathfrak{g}$ is simple and equipped with a Lie algebra grading. This Lie algebra grading corresponds to a group homomorphism $\mathbf{G}_\mathrm{m} \to G_\mathrm{ad}$, where G_ad is the adjoint group of $\mathfrak{g}$, that is to say the connected component of the identity in the group $\operatorname{Aut}_{\mathrm{Lie}}(\mathfrak{g})$ of Lie algebra automorphisms of $\mathfrak{g}$.

Using the fact that G_{ad} is isomorphic to the quotient of G by its centre, the conjugation action of G on itself induces an action of G_{ad} on G by group automorphisms. Composing this action with the homomorphism $\mathbf{G}_{\mathrm{m}} \to G_{\mathrm{ad}}$, we get a grading on G.

When N is graded, the semidirect product $N \rtimes \mathbf{G}_{\mathrm{m}}$ makes sense. A vector space V equipped with a an $N \rtimes \mathbf{G}_{\mathrm{m}}$-module structure is called a *graded N-module*. In particular, the $\mathbf{G}_{\mathrm{m}}$-action equips V with a $\mathbf{Z}$-grading. If N is connected, a graded N-module $V_\bullet$ is equivalently defined as an N-module V equipped with a $\mathbf{Z}$-grading such that, for any δ and p in $\mathbf{Z}$, $\mathfrak{n}_\delta \cdot V_p \subseteq V_{\delta+p}$.

Example 4.1.2.2. Let N be a graded algebraic group and X be an affine scheme equipped with $\mathbf{G}_{\mathrm{m}}$-action. If there is a $\mathbf{G}_{\mathrm{m}}$-equivariant action $\rho : N \times X \to X$, then $\rho^* : \mathbf{C}[X] \to \mathbf{C}[N] \otimes_{\mathbf{C}} \mathbf{C}[X]$ composed with the co-inverse makes $V_\bullet$ a graded N^--module.

In particular, if N is connected, the action ρ is $\mathbf{G}_{\mathrm{m}}$-equivariant if and only if for any δ and p in $\mathbf{Z}$, the inclusion $\mathfrak{n}_\delta \cdot \mathbf{C}[X]_p \subseteq \mathbf{C}[X]_{p-\delta}$ holds.

Lemma 4.1.2.3. *If N is graded and $V_\bullet$ is a graded N-module, then the subspaces $V_p^N := V^N \cap V_p$, for p in $\mathbf{Z}$, define a grading on V^N.*

Proof. According to Lemma 2.1.4.1 applied to the $N \ltimes \mathbf{G}_{\mathrm{m}}$-module V, the action of $\mathbf{G}_{\mathrm{m}}$ stabilises V^N. □

Notice that an N-module V is graded if V is a graded vector space and the comodule structure $\rho : V_\bullet \to \mathbf{C}[N]_\bullet \otimes_{\mathbf{C}} V_\bullet$ is a graded homomorphism. By analogy, we call a *filtered N-module* a filtered vector space $V_\bullet$ equipped with a co-action $\rho : V_\bullet \to \mathbf{C}[N]_\bullet \otimes_{\mathbf{C}} V_\bullet$ that is a filtered homomorphism.

Lemma 4.1.2.4. *If N is graded and $V_\bullet$ is a filtered N-module, then* $\operatorname{gr}_\bullet V$ *has an induced graded N-module structure.*

Proof. Take the graded homomorphism associated with the filtered co-action, it gives the graded N-module structure on $\operatorname{gr}_\bullet V$. □

Assume that N is connected. Recall that for an N-module V, the level 0 of the Lie algebra cohomology computes the N-invariants:

$$\mathrm{H}^0(\mathfrak{n}, V) = V^{\mathfrak{n}} = V^N.$$

The following proposition is a generalisation of [GG02, Proposition 5.2].

Proposition 4.1.2.5. *Let $\mathfrak{n} = \bigoplus_{\delta \in \mathbf{Z}_{<0}} \mathfrak{n}_\delta$ be a negatively graded finite-dimensional Lie algebra, N be the corresponding unipotent group and $V_\bullet$ be a filtered N-module. Make the following assumptions:*

1. *the subspaces V_p are zero for $p < 0$ and $V = \bigcup_{p \geqslant 0} V_p$,*

2. *there is an isomorphism $\mathbf{C}[N] \otimes_{\mathbf{C}} W \cong \operatorname{gr} V$ of N-modules, where $\mathbf{C}[N]$ is equipped with the left-multiplication action and W denotes a trivial N-module.*

Then there is a natural graded linear isomorphism

$$\operatorname{gr}_\bullet(V^N) \xrightarrow{\sim} (\operatorname{gr}_\bullet V)^N.$$

Moreover the Lie algebra cohomology $\mathrm{H}^n(\mathfrak{n}, V)$ is zero whenever $n \neq 0$.

Proof. Introduce the Chevalley–Eilenberg complex for the $\mathfrak{n}$-module V:

$$C^\bullet := V \otimes_{\mathbf{C}} \bigwedge\nolimits^\bullet \mathfrak{n}^*,$$

where $\bigwedge^\bullet \mathfrak{n}^*$ denotes the exterior algebra of $\mathfrak{n}^*$. Consider the dual $\mathbf{Z}$-grading on $\mathfrak{n}^*$, that is negative: $\mathfrak{n}^* := \bigoplus_{\delta \in \mathbf{Z}_>} \mathfrak{n}^*_\delta$. For p in $\mathbf{Z}$ and $n \geqslant 0$, define the linear subspace

$$\mathrm{F}_p\, C^n := \sum_{\substack{\ell, \delta_1, \dots, \delta_n \in \mathbf{Z} \\ \ell + \delta_1 + \cdots + \delta_n \leqslant p}} V_\ell \otimes_{\mathbf{C}} (\mathfrak{n}^*_{\delta_1} \wedge \cdots \wedge \mathfrak{n}^*_{\delta_n}).$$

This defines a filtration on the complex such that the differential verifies the inclusion $d(\mathrm{F}_p\, C^\bullet) \subseteq \mathrm{F}_p\, C^\bullet$ for all p in $\mathbf{Z}$. According to [Wei97, Section 5.4], there is a spectral sequence $\{(E_r, d_r)\}_{r=0}^\infty$ associated with this filtered cochain complex.

For p, q in $\mathbf{Z}$, the page 0 is given by:

$$E_0^{p,q} = \operatorname{gr}^{\mathrm{F}}_p C^{q-p} := \bigoplus_{\ell + \delta_1 + \cdots + \delta_{q-p} = p} \operatorname{gr}_\ell V \otimes_{\mathbf{C}} (\mathfrak{n}^*_{\delta_1} \wedge \cdots \wedge \mathfrak{n}^*_{\delta_{q-p}}).$$

It is the Chevalley–Eilenberg complex for the $\mathfrak{n}$-module $\operatorname{gr} V$: for $n \in \mathbf{Z}$, there is the identification

$$\bigoplus_{p \in \mathbf{Z}} E_0^{p,n+p} = \operatorname{gr} V \otimes_{\mathbf{C}} \bigwedge\nolimits^n \mathfrak{n}^*.$$

Hence the page 1 is the Lie algebra cohomology of $\operatorname{gr} V$:

$$\bigoplus_{p \in \mathbf{Z}} E_1^{p,n+p} = \mathrm{H}^n(\mathfrak{n}, \operatorname{gr} V).$$

By using the $\mathfrak{n}$-equivariant isomorphism $\mathbf{C}[N] \otimes_{\mathbf{C}} W \cong \operatorname{gr} V$, we get

$$\mathrm{H}^n(\mathfrak{n}, \operatorname{gr} V) \cong W \otimes_{\mathbf{C}} \mathrm{H}^n(\mathfrak{n}, \mathbf{C}[N]).$$

But the cohomology $\mathrm{H}^n(\mathfrak{n}, \mathbf{C}[N])$ coincides with the algebraic de Rham cohomology of N. Because N is isomorphic to the affine space $\mathbf{C}^{\dim N}$ (by the exponential map $\exp : \mathfrak{n} \xrightarrow{\sim} N$), its cohomology is given by

$$\mathrm{H}^n(\mathfrak{n}, \mathbf{C}[N]) \cong \delta_{n=0}\, \mathbf{C}.$$

Hence, the first page $E_1^{p,q}$ is zero unless $p = q$: the spectral sequence collapses.

Notice that the subspaces $\mathrm{F}_p C$ are zero for $p < 0$ and $\mathrm{F}_p C = \bigcup_{p \geqslant 0} \mathrm{F}_p C$. Moreover the spectral sequence is regular because it collapses. Hence, according to [Wei97, Theorem 5.5.10], the spectral sequence weakly converges to the Lie algebra cohomology of V: the infinity page is given by

$$E_\infty^{p,q} = \mathrm{gr}_p^{\mathrm{F}} \mathrm{H}^{q-p}(C^\bullet).$$

Because of the collapsing we get $E_\infty^{p,q} = E_1^{p,q}$. In particular, taking $p = q$, we see that the natural embedding

$$\mathrm{gr}_\bullet(V^N) = \mathrm{gr}_\bullet(V^{\mathfrak{n}}) \longhookrightarrow (\mathrm{gr}_\bullet V)^{\mathfrak{n}} = (\mathrm{gr}_\bullet V)^N$$

induced by the inclusion $V^N \subseteq V$ is an isomorphism. □

4.1.3. Quantum Hamiltonian reduction

Let M, N be two affine algebraic groups such that M is a normal subgroup of N, and denote by $\mathfrak{m}, \mathfrak{n}$ their respective Lie algebras. Assume that N acts on an associative unital algebra $\mathcal{A}$ by algebra automorphisms. Assume the existence of an N-equivariant Lie algebra homomorphism $\widehat{\mu} : \mathfrak{m} \to \mathcal{A}$ fitting in the following commutative triangle:

(4.1.3.1)
$$\begin{tikzcd} \mathfrak{m} \arrow[rr, "\text{action}"] \arrow[dr, "\widehat{\mu}"'] & & \operatorname{Der} \mathcal{A} \\ & \mathcal{A}. \arrow[ur, "F \mapsto [F, \bullet]"'] & \end{tikzcd}$$

The map $\widehat{\mu}$ is called a *quantum comoment map.*

Denote by $\widehat{I}$ the left ideal of $\mathcal{A}$ spanned by the $\widehat{\mu}(x)$ for x in $\mathfrak{m}$. The N-action stabilises the ideal $\widehat{I}$ and descends to the quotient $\mathcal{A}/\widehat{I}$, that is a left $\mathcal{A}$-module.

Lemma 4.1.3.2. *The invariant subspace $(\mathcal{A}/\widehat{I})^N$ is an associative algebra for the multiplication defined for $F_1 \bmod \widehat{I}, F_2 \bmod \widehat{I} \in (\mathcal{A}/\widehat{I})^N$ by the formula*

$$(F_1 \bmod \widehat{I}) \cdot (F_2 \bmod \widehat{I}) \coloneqq (F_1 \cdot F_2) \bmod \widehat{I}.$$

The algebra $(\mathcal{A}/\widehat{I})^N$ is called *quantum Hamiltonian reduction.*

Proof. Introduce the associative subalgebra

$$\mathcal{B} \coloneqq \{F \in \mathcal{A} \mid \text{for all } g \in N,\ g \cdot F - F \in I\}.$$

We need to prove that $\mathcal{B}$ is an algebra and that $\widehat{I}\mathcal{B}$ is included in $\widehat{I}$, so $\widehat{I}$ is a two-sided ideal of $\mathcal{B}$ and $(\mathcal{A}/\widehat{I})^N = \mathcal{B}/\widehat{I}$ is an algebra. The proof is then analogous to the proof of Lemma 2.1.2.5, with the bracket $[\bullet, \bullet]$ playing the role of the Poisson bracket $\{\bullet, \bullet\}$. □

Assume that $\mathfrak{n} = \bigoplus_{\delta \in \mathbf{Z}_{<0}} \mathfrak{n}_\delta$ is equipped with a negative Lie algebra grading and that $\mathfrak{m}$ is a graded subalgebra. Moreover, assume that N and M connected. Let $\mathcal{A}_\bullet$ be an almost commutative algebra of degree $-k$. By definition, $\mathcal{A}_\bullet$ is a quantisation of the affine Poisson scheme $X \coloneqq \operatorname{Spec}(\operatorname{gr} \mathcal{A})$.

Consider an action of N acts on $\mathcal{A}_\bullet$ by algebra automorphisms with a quantum comoment map $\widehat{\mu} : \mathfrak{m} \to \mathcal{A}$. We say that the quantum comoment map is *of degree* k if for any δ in $\mathbf{Z}$ and for any x in $\mathfrak{m}_\delta$, $\widehat{\mu}(x)$ belongs to $\mathcal{A}_{k+\delta}$. In this case, the algebra $\mathcal{A}_\bullet$ is a filtered N-module. According to Lemma 4.1.2.4, there is an induced action of N on X by Poisson automorphisms and $\mathbf{C}[X]_\bullet$ is a graded N-module.

Lemma 4.1.3.3. *Assume that $\widehat{\mu} : \mathfrak{m} \to \mathcal{A}$ is of degree k. Then there is an induced N-equivariant momemt map $\mu : X \to \mathfrak{m}^*$ whose comorphism is the graded map $\operatorname{gr} \widehat{\mu}$ associated with the quantum comoment map.*

Denote by $\widehat{I}$ the left ideal of $\mathcal{A}$ spanned by the $\widehat{\mu}(x)$ for x in $\mathfrak{m}$. It has an induced filtration defined for p in $\mathbf{Z}$ by $\widehat{I}_p \coloneqq \widehat{I} \cap \mathcal{A}_p$. The quotient $\mathcal{A}/\widehat{I}$ has an induced filtration too, defined by $(\mathcal{A}/\widehat{I})_p \coloneqq \mathcal{A}_p/\widehat{I}_p$. Hence there is an exact sequence

$$0 \longrightarrow \widehat{I}_\bullet \longrightarrow \mathcal{A}_\bullet \longrightarrow (\mathcal{A}/\widehat{I})_\bullet \longrightarrow 0$$

of filtered homomorphisms. The quantum Hamiltonian reduction $(\mathcal{A}/\widehat{I})^N$ has an induced filtration defined by

$$\left((\mathcal{A}/\widehat{I})^N\right)_p \coloneqq (\mathcal{A}/\widehat{I})_p \cap (\mathcal{A}/\widehat{I})^N \quad \text{for} \quad p \in \mathbf{Z}.$$

The following lemma is well-known, see for instance [ACET24, Lemma 2.3].

Lemma 4.1.3.4. *The induced graded homomorphism sequence*

$$0 \longrightarrow \operatorname{gr}_\bullet \widehat{I} \longrightarrow \operatorname{gr}_\bullet \mathcal{A} \longrightarrow \operatorname{gr}_\bullet(\mathcal{A}/\widehat{I}) \longrightarrow 0 \tag{4.1.3.5}$$

is exact.

Under the assumptions of Lemma 4.1.3.3, introduce I the left ideal of $\mathbf{C}[X]$ spanned by the $\mu^*(x)$ for x in $\mathfrak{m}$. Hence there is an exact sequence

$$0 \longrightarrow I_\bullet \longrightarrow \mathbf{C}[X]_\bullet \longrightarrow \mathbf{C}[\mu^{-1}(0)]_\bullet \longrightarrow 0 \tag{4.1.3.6}$$

of graded homomorphisms. The graded ring $\mathbf{C}[\mu^{-1}(0)]_\bullet$ is a graded N^{op}-module. Hence, Lemma 4.1.2.3 implies that $\mathbf{C}[\mu^{-1}(0)/\!/N]$ is a graded subring of $\mathbf{C}[\mu^{-1}(0)]_\bullet$.

Lemma 4.1.3.7. *The exact sequences* (4.1.3.5) *and* (4.1.3.6) *of graded homomorphisms are isomorphic.*

Proof. Because the comorphism μ^* identifies to $\operatorname{gr}\mu$ through the graded algebra isomorphism $\mathbf{C}[X]_\bullet \cong \operatorname{gr}_\bullet \mathcal{A}$, this isomorphism identifies the ideal $I_\bullet$ to $\operatorname{gr}_\bullet \widehat{I}$, and the quotient are automatically identified. $\square$

The following theorem generalises [GG02, Theorem 4.1].

Theorem 4.1.3.8. *Let N be a unipotent algebraic group such that its Lie algebra $\mathfrak{n} = \bigoplus_{\delta \in \mathbf{Z}_{<0}} \mathfrak{n}_\delta$ is negatively graded. Let $\mathcal{A}_\bullet$ be an almost commutative algebra of degree $-k$ with an action of N by algebra automorphisms and let $\widehat{\mu} : \mathfrak{m} \to \mathcal{A}$ be an N-equivariant quantum comoment map of degree k. Define the affine Poisson scheme $X := \operatorname{Spec}(\operatorname{gr}_\bullet A)$ and the moment map $\mu : X \to \mathfrak{m}^*$ such that $\operatorname{gr}\widehat{\mu} = \mu^*$. Make the following assumptions:*

1. *the induced filtration on the quotient $\mathcal{A}/\widehat{I}$ is nonnegative,*
2. *there is a closed subscheme S of $\mu^{-1}(0)$ such that the action map*
$$\alpha : N \times S \longrightarrow \mu^{-1}(0), \quad (g, x) \longmapsto g \cdot x$$
is an isomorphism.

Then there is a natural graded Poisson isomorphism

$$\operatorname{gr}_\bullet \left((\mathcal{A}/\widehat{I})^N \right) \xrightarrow{\sim} \mathbf{C}[\mu^{-1}(0)/\!/N]_\bullet,$$

Proof. First, there is a graded algebra isomorphism $\operatorname{gr}_\bullet(\mathcal{A}/\widehat{I}) \cong \mathbf{C}[\mu^{-1}(0)]_\bullet$ that is N-equivariant by Lemma 4.1.3.7, hence we have an graded algebra isomorphism

$$\big(\operatorname{gr}_\bullet(\mathcal{A}/\widehat{I})\big)^N \cong \mathbf{C}[\mu^{-1}(0)]_\bullet^N.$$

We have an N-module isomorphism $\mathbf{C}[\mu^{-1}(0)] \cong \mathbf{C}[N] \otimes_{\mathbf{C}} \mathbf{C}[S]$. By Proposition 4.1.2.5, we get that the natural graded algebra embedding

$$\operatorname{gr}_\bullet(\mathcal{A}/\widehat{I})^N \longhookrightarrow \big(\operatorname{gr}_\bullet(\mathcal{A}/\widehat{I})\big)^N$$

is an isomorphism.

By composing the two previous maps, we get an algebra isomorphism given for p in $\mathbf{Z}$ by

$$\operatorname{gr}_\bullet(\mathcal{A}/\widehat{I})^N \xrightarrow{\sim} \mathbf{C}[\mu^{-1}(0)]_\bullet^N, \quad [a \bmod \widehat{I}]_p \longmapsto [a]_p \bmod I,$$

and by definitions of the Poisson bracket on the left-hand side and on the Hamiltonian reduction, this isomorphism is compatible with the Poisson structures. $\square$

4.1.4. Quantum reduction by stages

It is a natural analogue of the construction from Section 2.1.4. Consider a linear algebraic group N_2 and N_1, N_0 two closed subgroups such that N_1 is a normal subgroup and the semidirect product decomposition $N_2 = N_1 \rtimes N_0$ holds.

Let $\mathcal{A}$ be an associative unital algebra with an action of N_2 by algebra automorphisms and an N_2-equivariant quantum moment map $\widehat{\mu}_2 : \mathfrak{n}_2 \to \mathcal{A}$. The restriction of this quantum comoment map to $\mathfrak{n}_1$, denoted by $\widehat{\mu}_1 : \mathfrak{n}_1 \to \mathcal{A}$ is a quantum comoment map for the N_1-action obtained by restriction, and it is N_2-equivariant. For $i = 1, 2$, denote by $\widehat{I}_i$ the left ideal of $\mathcal{A}$ spanned by the $\widehat{\mu}_i(x)$ for x in $\mathfrak{n}_i$.

The action of N_2 descends to $A/\widehat{I}_1$ because the ideal is stabilised by the N_2-action, since $\widehat{\mu}_2$ is N_2-equivariant. According to Lemma 2.1.4.1, there is an induced action of N_0 on $(A/\widehat{I}_1)^{N_1}$.

Lemma 4.1.4.1. *The induced action of N_0 on $(A/\widehat{I}_1)^{N_1}$ has a quantum comoment map $\widehat{\mu}_0 : \mathfrak{n}_0 \to (\mathcal{A}/\widehat{I}_1)^{N_1}$ induced by the projection of $\widehat{\mu}_2$ to $\mathcal{A}/\widehat{I}_1$.*

Proof. By restriction and projection, we can define a map $\widehat{\mu}_0 : \mathfrak{n}_0 \to \mathcal{A}/\widehat{I}_1$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathfrak{n}_2 \arrow[r, "\widehat{\mu}_2"] & \mathcal{A} \arrow[d] \\
\mathfrak{n}_0 \arrow[u, hook] \arrow[r, "\widehat{\mu}_0"'] & A/\widehat{I}_1.
\end{tikzcd}$$

Explicitely, for any x in $\mathfrak{n}_0$, one has

$$\widehat{\mu}_0(x) \coloneqq \widehat{\mu}_2(x) \bmod \widehat{I}_1.$$

We claim that for any g in N_1, $g \cdot \widehat{\mu}_0(x) = \widehat{\mu}_0(x)$. It is equivalent to prove that $g \cdot \widehat{\mu}_2(x) - \widehat{\mu}_2(x) = \widehat{\mu}_2(\mathrm{Ad}(g)x - x)$ belongs to $\widehat{I}_1$. This follows from the fact that $\mathrm{Ad}(g)x - x$ belongs to $\mathfrak{n}_1$, see the proof of Lemma 2.1.4.2. So we get the desired map $\widehat{\mu}_0 : \mathfrak{n}_0 \to (\mathcal{A}/\widehat{I}_1)^{N_1}$. Proving that it is a comoment map is done in the same way as in the proof of Lemma 2.1.4.3. $\square$

Denote by $\widehat{I}_0$ the left ideal of $(\mathcal{A}/\widehat{I}_1)^{N_1}$ spanned by the $\widehat{\mu}_0(x)$ for x in $\mathfrak{n}_0$.

Proposition 4.1.4.2. *The projection $\mathcal{A}/\widehat{I}_1 \twoheadrightarrow \mathcal{A}/\widehat{I}_2$ induces a well defined N_0-equivariant homomorphism*

$$(\mathcal{A}/\widehat{I}_1)^{N_1}/\widehat{I}_0 \longrightarrow (\mathcal{A}/\widehat{I}_2)^{N_1}. \tag{4.1.4.3}$$

It induces an algebra homomorphism between the Hamiltonian reductions:

$$\left((\mathcal{A}/\widehat{I}_1)^{N_1}/\widehat{I}_0\right)^{N_0} \longrightarrow (\mathcal{A}/\widehat{I}_2)^{N_2}. \tag{4.1.4.4}$$

If (4.1.4.3) *is an isomorphism, then so is* (4.1.4.4).

Figure 4.1: Quantum reduction by stages homomorphism

$$\begin{tikzcd}
\mathcal{A}/\widehat{I}_1 \arrow[r, two heads] & \mathcal{A}/\widehat{I}_2 \\
(\mathcal{A}/\widehat{I}_1)^{N_1} \arrow[u, hook] \arrow[r] \arrow[d, two heads] & (\mathcal{A}/\widehat{I}_2)^{N_1} \arrow[u, hook] \\
(\mathcal{A}/\widehat{I}_1)^{N_1}/I_0 \arrow[ur] & \\
\big((\mathcal{A}/\widehat{I}_1)^{N_1}/I_0\big)^{N_0} \arrow[u, hook] \arrow[r] & (\mathcal{A}/\widehat{I}_2)^{N_2} \arrow[uu, hook]
\end{tikzcd}$$

Proof. Because of the N_1-equivariance, the projection induces a homomorphism

$$(\mathcal{A}/\widehat{I}_1)^{N_1}/\widehat{I}_0 \longrightarrow (\mathcal{A}/\widehat{I}_2)^{N_1},$$

and this homomorphisms maps clearly $\widehat{I}_0$ to 0, so (4.1.4.3) is well-defined. Taking the N_0-invariant and using Lemma 2.1.4.1, one gets (4.1.4.4). The last part of the statement is clear. □

The construction of this section is summarized in Figure 4.1.

4.2. Construction of finite W-algebras

We give a definition of the finite W-algebras $\mathcal{U}(\mathfrak{g}, f)$ as quantum Hamiltonian reduction, following the original idea of Premet [Pre02] but in a formulation inspired by [DDCDS+06] (where they consider Hamiltonian reduction with respect to a nontrivial coadjoint orbit). We recall the construction of the Kazhdan filtration on $\mathcal{U}(\mathfrak{g}, f)$ and the proof of the Poisson isomorphism between the graded algebra $\operatorname{gr}\mathcal{U}(\mathfrak{g}, f)$ and $\mathbf{C}[S_f]$ (Theorem 4.2.3.1), due to [GG02]. We give an elementary construction of a PBW basis on this W-algebra (Corollary 4.2.3.5).

4.2.1. Kazhdan filtration

Keep the notations from Section 2.2: $\mathfrak{g}$ is a simple finite-dimensional complex Lie algebra, $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ and G be a connected algebraic group whose Lie algebra is $\mathfrak{g}$. Denote by $(\bullet|\bullet) := (2h^\vee)^{-1}\kappa_{\mathfrak{g}}$ the normalised non-degenerate symmetric bilinear form on $\mathfrak{g}$. Let f a nilpotent element of $\mathfrak{g}$ and $\mathfrak{g} = \bigoplus_{\delta \in \mathbf{Z}} \mathfrak{g}_\delta$ be a good grading for f defined by the adjoint action of a semisimple element H in $\mathfrak{h}$.

Let $\mathcal{U}(\mathfrak{g})$ be the universal enveloping algebra of the simple Lie algebra $\mathfrak{g}$. Its PBW filtration is defined by

$$\mathcal{U}(\mathfrak{g})_p^{\mathrm{PBW}} \coloneqq \sum_{n=0}^{n} \{x_1 \cdots x_n \mid x_1, \ldots, x_n \in \mathfrak{g}\}$$

for $p \geqslant 0$ an integer.

The good grading induces a filtration on $\mathcal{U}(\mathfrak{g})$, called the *Kazhdan filtration*, that is defined for p in $\mathbf{Z}$ by:

$$\begin{aligned}\mathcal{U}(\mathfrak{g})_p &\coloneqq \sum_{2n-\delta \leqslant p} \{F \in \mathcal{U}(\mathfrak{g})_n^{\mathrm{PBW}} \mid [H, F] = \delta\, F\} \\ &= \bigoplus_{\delta \in \mathbf{Z}} \{F \in \mathcal{U}(\mathfrak{g})_{\lfloor \frac{p+\delta}{2} \rfloor}^{\mathrm{PBW}} \mid [H, F] = \delta\, F\}.\end{aligned}$$

The natural embedding $\mathfrak{g} \subseteq \mathcal{U}(\mathfrak{g})$ induces an embedding $\mathfrak{g}_j \subseteq \mathcal{U}(\mathfrak{g})_{2-\delta}$ for δ in $\mathbf{Z}$.

Let $\{x_1, \ldots, x_r\}$ be an ordered basis of $\mathfrak{g}$ that is homogenous for the good grading and call $\delta(x_i)$ the degree of x_i. By the Poincaré–Birkhoff–Witt Theorem [Hum72, Corollary 17.3.C], the ordered products

$$\{x_{i_1} \cdots x_{i_k} \mid k \geqslant 0,\ 1 \leqslant i_1 \leqslant \cdots \leqslant i_k \leqslant r\}$$

form a basis of $\mathcal{U}(\mathfrak{g})$, called *PBW basis*.

Lemma 4.2.1.1. *For for p in $\mathbf{Z}$, $\mathcal{U}(\mathfrak{g})_p$ admits for basis the subfamily*

$$\{x_{i_1} \cdots x_{i_k} \mid 2k - (\delta(x_{i_1}) + \cdots + \delta(x_{i_k})) \leqslant p\}.$$

Proof. It is clear that theses products belong to $\mathcal{U}(\mathfrak{g})_p$ because H acts by derivation, so

$$[H, x_{i_1} \cdots x_{i_k}] = (\delta(x_{i_1}) + \cdots + \delta(x_{i_k}))\, x_{i_1} \cdots x_{i_k}$$

and $\delta \coloneqq \delta(x_1) + \cdots + \delta(x_k)$ satisfies $2k - \delta \leqslant p$. To conclude, we need to check they span everything.

Take a generator Y of $\mathcal{U}(\mathfrak{g})_p$. By definition, $Y = y_1 \cdot y_n$ is a product of elements y_i in $\mathcal{U}(\mathfrak{g})$ and if δ is in $\mathbf{Z}$ such that $2n - \delta \leqslant p$ and $[H, Y] = \delta Y$. Write Y in the PBW basis

$$Y = \sum_{1 \leqslant i_1 \leqslant \cdots \leqslant i_k \leqslant r} c_{i_1, \ldots, i_k} x_{i_1} \cdots x_{i_k}, \quad \text{where} \quad c_{i_1, \ldots, i_k} \in \mathbf{C}.$$

Applying the bracket of H on both sides and identifying, we see that the right-hand side sum must be H-homogeneous of degree δ, hence

$$Y = \sum_{\delta(x_{i_1}) + \cdots + \delta(x_{i_k}) = \delta} c_{i_1, \ldots, i_k} x_{i_1} \cdots x_{i_k}$$

is a linear combination of the desired products. $\square$

Because the good grading is a Lie algebra grading and $\mathfrak{g}$ spans $\mathcal{U}(\mathfrak{g})$ as an algebra, we easily check the inclusion

$$[\mathcal{U}(\mathfrak{g})_p, \mathcal{U}(\mathfrak{g})_q] \subseteq \mathcal{U}(\mathfrak{g})_{p+q-2} \quad \text{for} \quad p, q \in \mathbf{Z}.$$

According to Section 4.1.1, the filtered algebra $\mathcal{U}(\mathfrak{g})$ is almost commutative of degree -2 and the associated graded algebra $\operatorname{gr}_\bullet \mathcal{U}(\mathfrak{g})$ is commutative and Poisson.

The definition of this filtration is related to the $\mathbf{G}_\mathrm{m}$-action that is defined in (2.2.3.5), $\rho : \mathbf{G}_\mathrm{m} \to \operatorname{Aut}_\mathbf{C}(\mathfrak{g}^*)$, and corresponds to a co-action

$$\rho^* : \mathbf{C}[\mathfrak{g}^*] \longrightarrow \mathbf{C}[t, t^{-1}] \otimes_\mathbf{C} \mathbf{C}[\mathfrak{g}^*].$$

The *Kazhdan grading* on $\mathbf{C}[\mathfrak{g}^*] = \operatorname{Sym} \mathfrak{g}$ is defined for all p in $\mathbf{Z}$ by

$$\mathbf{C}[\mathfrak{g}^*]_p \coloneqq \{F \in \mathbf{C}[\mathfrak{g}^*] \mid \rho^*(F) = t^p \otimes F\}.$$

Any element x in $\mathfrak{g}_\delta$, for δ in $\mathbf{Z}$, belongs to $\mathbf{C}[\mathfrak{g}^*]_{2-\delta}$, so we deduce that

$$\mathbf{C}[\mathfrak{g}^*]_p = \bigoplus_{2n-\delta=p} \{x_1 \cdots x_n \mid x_1, \dots, x_n \in \mathfrak{g},\ \{H, x_1 \cdots x_n\} = \delta\, x_1 \cdots x_n\}.$$

By the same kind of argumentq as for Lemma 4.2.1.1, we see that if $\{x_1, \dots, x_r\}$ is a homogeneous basis of $\mathfrak{g}$, where $\delta(x_i)$ denotes the degree of x_i, then the ordered products

$$\left\{ x_{i_1} \cdots x_{i_k} \;\middle|\; \begin{array}{l} k \geqslant 0,\ 1 \leqslant i_1 \leqslant \cdots \leqslant i_k \leqslant r \\ 2k - (\delta(x_{i_1}) + \cdots + \delta(x_{i_k})) = p \end{array} \right\}$$

form a basis of $\mathbf{C}[\mathfrak{g}^*]_p$.

Proposition 4.2.1.2 ([GG02])**.** *The linear map $\mathfrak{g} \to \operatorname{gr} \mathcal{U}(\mathfrak{g})$ sending x in $\mathfrak{g}_\delta$ to $[x]_{2-\delta}$, for δ in $\mathbf{Z}$ induces a graded Poisson algebra isomorphism*

$$\mathbf{C}[\mathfrak{g}^*]_\bullet \cong \operatorname{gr}_\bullet \mathcal{U}(\mathfrak{g}).$$

Hence $\mathcal{U}(\mathfrak{g})$ is a quantisation of $\mathfrak{g}^*$.

Proof. It is clear that the mapping defines a Lie algebra map $\mathfrak{g} \to \operatorname{gr}_\bullet \mathcal{U}(\mathfrak{g})$, where the right-hand side is equipped with the Poisson bracket coming from the filtration. Then there is a graded Poisson algebra homomorphism

$$\mathbf{C}[\mathfrak{g}^*]_\bullet \longrightarrow \operatorname{gr}_\bullet \mathcal{U}(\mathfrak{g}). \tag{4.2.1.3}$$

To check it is an isomorphism, we use Lemma 4.2.1.1. Let $\{x_1, \dots, x_r\}$ be a homogeneous basis of $\mathfrak{g}$, where $\delta(x_i)$ denotes the degree of x_i. For p in $\mathbf{Z}$, the lemma implies that $\operatorname{gr}_p \mathcal{U}(\mathfrak{g})$ admits for basis the products

$$\left\{ [x_{i_1} \cdots x_{i_k}]_p \;\middle|\; \begin{array}{l} k \geqslant 0,\ 1 \leqslant i_1 \leqslant \cdots \leqslant i_k \leqslant r \\ 2k - (\delta(x_{i_1}) + \cdots + \delta(x_{i_k})) = p \end{array} \right\}$$

Because of the equality

$$[x_{i_1}\cdots x_{i_k}]_p = [x_{i_1}]_{2-\delta(x_{i_1})}\cdots[x_{i_k}]_{2-\delta(x_{i_k})},$$

it is now clear that the map (4.2.1.3) is an isomorphism because it map a basis of $\mathbf{C}[\mathfrak{g}^*]_p$ to a basis of $\operatorname{gr}_p \mathcal{U}(\mathfrak{g})$. $\square$

4.2.2. Quantum Hamiltonian reduction

Let $\mathfrak{l}$ be an isotropic subspace of the symplectic space $(\mathfrak{g}_1, \omega)$. Let $N_{\mathfrak{l}}$ be the unipotent subgroup of G whose Lie algebra is $\mathfrak{n}_{\mathfrak{l}} = \mathfrak{l}^{\perp,\omega}$. Denote by $\chi = (f|\bullet)$ the linear form on $\mathfrak{g}$ associated with f and by $\overline{\chi}_{\mathfrak{l}}$ its restriction to $\mathfrak{n}_{\mathfrak{l}}$. Recall that there is a symplectic isomorphism $\sigma_l : \mathfrak{l}^{\perp,\omega}/\mathfrak{l} \to \mathcal{O}_{\mathfrak{l}}$, where $\mathcal{O}_{\mathfrak{l}} = \operatorname{Ad}^*(N_{\mathfrak{l}})\overline{\chi}_{\mathfrak{l}}$.

The group $N_{\mathfrak{l}}$ acts on $\mathcal{U}(\mathfrak{g})$ by the adjoint action. It also acts on $\mathfrak{l}^{\perp,\omega}/\mathfrak{l}$ by symplectic automorphisms, so its acts on the Weyl algebra $\mathcal{W}(\mathfrak{l}^{\perp,\omega}/\mathfrak{l})$ that was defined in Example 4.1.1.2. We get a diagonal action on the tensor product $\mathcal{U}(\mathfrak{g}) \otimes_{\mathbf{C}} \mathcal{W}(\mathfrak{l}^{\perp,\omega}/\mathfrak{l})$ by algebra automorphisms.

It is not difficult to check that this action has a quantum comoment map

$$\widehat{\mu}_{\mathfrak{l}} : \mathfrak{n}_{\mathfrak{l}} \longrightarrow \mathcal{U}(\mathfrak{g}) \otimes_{\mathbf{C}} \mathcal{W}(\mathfrak{l}^{\perp,\omega}/\mathfrak{l})$$

defined for any element x in $\mathfrak{n}_{\mathfrak{l}}$ by

$$\widehat{\mu}_{\mathfrak{l}}(x) \coloneqq \begin{cases} x \otimes 1 + 1 \otimes (x \bmod \mathfrak{l}) & \text{if} \quad x \in \mathfrak{l}^{\perp,\omega}, \\ x \otimes 1 + \chi(x)1 & \text{if} \quad x \in \mathfrak{g}_2, \\ x \otimes 1 & \text{otherwise.} \end{cases} \tag{4.2.2.1}$$

Let $\widehat{I}_{\mathfrak{l}}$ be the left ideal of $\mathcal{U}(\mathfrak{g}) \otimes_{\mathbf{C}} \mathcal{W}(\mathfrak{l}^{\perp,\omega}/\mathfrak{l})$ spanned by the $\widehat{\mu}_{\mathfrak{l}}(x)$ for x in $\mathfrak{n}_{\mathfrak{l}}$ and set

$$\mathcal{Q}_l \coloneqq \big(\mathcal{U}(\mathfrak{g}) \otimes_{\mathbf{C}} \mathcal{W}(\mathfrak{l}^{\perp,\omega}/\mathfrak{l})\big)/\widehat{I}_{\mathfrak{l}}$$

the corresponding quotient left module.

Definition 4.2.2.2 ([Pre02, GG02])**.** *The **finite W-algebra** associated with $\mathfrak{g}$ and f is defined as the quantum Hamiltonian reduction corresponding to the comoment map $\widehat{\mu}$ and is denoted by:*

$$\mathcal{U}(\mathfrak{g}, f) \coloneqq {\mathcal{Q}_l}^{N_{\mathfrak{l}}}.$$

Remark 4.2.2.3. Up to algebra isomorphism, this definition only depends on the nilpotent orbit $\mathbf{O}$ containing f [BG07, Section 4 and 5]. This can be proved by using the results from Section 2.2.5 and the Kazhdan filtration described below, see [GG02, Section 5.5] for a proof.

In Section 6.1.2, we prove the equivalence of the definitions for affine W-algebras by using an *intermediary orbit*. Note that this proof has an obvious analogue for finite W-algebra.

Introduce the filtration on $\mathscr{W}(\mathfrak{l}^{\perp,\omega}/\mathfrak{l})$ that is defined for p in $\mathbf{Z}$ by:

$$\mathscr{W}(\mathfrak{l}^{\perp,\omega}/\mathfrak{l})_p \coloneqq \sum_{n\leqslant p}\{v_1\cdots v_n \mid v_1,\dots,v_n \in \mathfrak{l}^{\perp,\omega}/\mathfrak{l}\}.$$

In particular, $\mathfrak{l}^{\perp,\omega}/\mathfrak{l}$ is naturally included in $\mathscr{W}(\mathfrak{l}^{\perp,\omega}/\mathfrak{l})_1$. By tensor product, we get an induced Kazhdan filtration on $\mathcal{U}(\mathfrak{g})\otimes_{\mathbf{C}}\mathscr{W}(\mathfrak{l}^{\perp,\omega}/\mathfrak{l})$ defined by

$$\big(\mathcal{U}(\mathfrak{g})\otimes_{\mathbf{C}}\mathscr{W}(\mathfrak{l}^{\perp,\omega}/\mathfrak{l})\big)_p \coloneqq \sum_{i+j\leqslant p}\mathcal{U}(\mathfrak{g})_i\otimes_{\mathbf{C}}\mathscr{W}(\mathfrak{l}^{\perp,\omega}/\mathfrak{l})_j \quad \text{for} \quad p\in\mathbf{Z},$$

The good grading induces a grading on $N_{\mathfrak{l}}$. Moreover, we have the inclusion

$$\widehat{\mu}(\mathfrak{g}_\delta)\subseteq(\mathcal{U}(\mathfrak{g})\otimes_{\mathbf{C}}\mathscr{W}(\mathfrak{l}^{\perp,\omega}/\mathfrak{l}))_{2-\delta}$$

for any δ in $\mathbf{Z}$. So by Section 4.1.2, $\mathcal{U}(\mathfrak{g})\otimes_{\mathbf{C}}\mathscr{W}(\mathfrak{l}^{\perp,\omega}/\mathfrak{l})$ is a filtered ${N_{\mathfrak{l}}}^-$-module, where the "$-$" supscript means that we are considering the opposite grading of the good grading (see Section 4.1.2 for the definitions). The ideal $\widehat{I}_{\mathfrak{l}}$, the quotient $\mathcal{Q}_{\mathfrak{l}}$ and then the Hamiltonian reduction $\mathcal{U}(\mathfrak{g},f)$ have an induced Kazhdan filtration.

4.2.3. Quantisation of the Slodowy slice

In Section 2.2.3, we introduced the moment map

$$\mu_{\mathfrak{l}}:\mathfrak{g}^*\times(\mathfrak{l}^{\perp,\omega}/\mathfrak{l})\longrightarrow{\mathfrak{n}_{\mathfrak{l}}}^*,\quad(\xi,v \bmod \mathfrak{l})\longmapsto\pi_{\mathfrak{l}}(\xi)+\overline{\chi}_{\mathfrak{l}}+\operatorname{ad}^*(v)\overline{\chi}_{\mathfrak{l}},$$

where $\pi_{\mathfrak{l}}:\mathfrak{g}^*\twoheadrightarrow{\mathfrak{n}_{\mathfrak{l}}}^*$ is the restriction map. If we equip $\mathfrak{l}^{\perp,\omega}/\mathfrak{l}$ with the $\mathbf{G}_{\mathrm{m}}$-action by scalar multiplication, $\mathfrak{g}^*$ and ${\mathfrak{n}_{\mathfrak{l}}}^*$ with the action ρ, then this moment map is $\mathbf{G}_{\mathrm{m}}$-equivariant and ${\mu_{\mathfrak{l}}}^{-1}(0)$ is a $\mathbf{G}_{\mathrm{m}}$-stable subvariety.

As a direct consequence of Lemma 2.2.3.6 and Example 4.1.2.2 $\mathbf{C}[{\mu_{\mathfrak{l}}}^{-1}(0)]_\bullet$ is equipped with a graded ${N_{\mathfrak{l}}}^-$-module structure. Hence,

$$\mathscr{P}(\mathfrak{g},f)=\mathbf{C}[{\mu_{\mathfrak{l}}}^{-1}(0)]^{N_{\mathfrak{l}}}$$

is a graded subspace by Lemma 4.1.2.3. The Poisson bracket of $\mathbf{C}[\mathfrak{g}^*]\otimes_{\mathbf{C}}[\mathfrak{l}^{\perp,\omega}/\mathfrak{l}]$ is of degree -2, so the same holds for the Poisson bracket of $\mathscr{P}(\mathfrak{g},f)_\bullet$.

Theorem 4.2.3.1 ([GG02, Theorem 4.1]). *There is a graded Poisson algebra isomorphism*

$$\mathscr{P}(\mathfrak{g},f)_\bullet\cong\operatorname{gr}_\bullet\mathcal{U}(\mathfrak{g},f).$$

Because $\mathscr{P}(\mathfrak{g},f)$ is isomorphic to the coordinate ring of the Slodowy slice S_f, the finite W-algebra is a quantisation of the slice.

The proof of Theorem 4.2.3.1 in [GG02] is a particular case of the proof of Theorem 4.1.3.8. Let us explain why it applies here.

According to Example 4.1.1.2 there is a graded Poisson isomorphism and then there is an induced graded isomorphism

$$\mathbf{C}[\mathfrak{g}^*]_\bullet\otimes_{\mathbf{C}}\mathbf{C}[\mathfrak{l}^{\perp,\omega}/\mathfrak{l}]_\bullet\cong\operatorname{gr}_\bullet\big(\mathcal{U}(\mathfrak{g})\otimes_{\mathbf{C}}\mathscr{W}(\mathfrak{l}^{\perp,\omega}/\mathfrak{l})\big).\tag{4.2.3.2}$$

Lemma 4.2.3.3. *The graded map* $\operatorname{gr}\widehat{\mu}_{\mathfrak{l}}$ *associated with the comoment map identifies with the comorphism of the twisted moment map* $\mu_{\mathfrak{l}}$*. Hence there is a commutative graded algebra isomorphism*

$$\mathbf{C}[{\mu_l}^{-1}(0)]_\bullet \cong \operatorname{gr}_\bullet \mathcal{Q}_{\mathfrak{l}}$$

induced by the isomorphism (4.2.3.2)*.*

Proof. The identification ${\mu_{\mathfrak{l}}}^* = \operatorname{gr}\widehat{\mu}_{\mathfrak{l}}$ is clear by comparing their formulae. The isomorphism is a consequence of the exact sequence isomorphism given by Lemma 4.1.3.7 applied to this quantised Hamiltonian reduction. □

Lemma 4.2.3.4. *The ideal* $\widehat{I}$ *contains* $(\mathcal{U}(\mathfrak{g}) \otimes_{\mathbf{C}} \mathcal{W}(\mathfrak{l}^{\perp,\omega}/\mathfrak{l}))_{-1}$*. It follows that* $\mathcal{Q}_{\mathfrak{l}}$ *and* $\mathcal{U}(\mathfrak{g}, f)$ *are nonnegatively filtered:* $(\mathcal{Q}_{\mathfrak{l}})_{-1} = 0$ *and* $\mathcal{U}(\mathfrak{g}, f)_{-1} = 0$*.*

Proof. The algebra $\mathcal{U}(\mathfrak{g}) \otimes_{\mathbf{C}} \mathcal{W}(\mathfrak{l}^{\perp,\omega}/\mathfrak{l})$ is generated by the subspaces $\mathfrak{g}_\delta$ for δ in $\mathbf{Z}$ and $\mathfrak{l}^{\perp,\omega}/\mathfrak{l}$. Note that $(\mathcal{U}(\mathfrak{g}) \otimes_{\mathbf{C}} \mathcal{W}(\mathfrak{l}^{\perp,\omega}/\mathfrak{l}))_{-1}$ is contained in the left-ideal spanned by $\mathfrak{g}_{\geqslant 3}$ that is contained in $\widehat{\mu}_{\mathfrak{l}}(\mathfrak{n}_{\mathfrak{l}})$. After taking quotient by $\widehat{I}_{\mathfrak{l}}$, the negative part of the filtration is killed. □

We can conclude the proof of the main theorem.

Proof of Theorem 4.2.3.1. Lemma 4.2.3.4 and the isomorphism

$$N_{\mathfrak{l}} \times S_f \cong {\mu_{\mathfrak{l}}}^{-1}(0)$$

given by Theorem 2.2.3.3 allow us to apply Theorem 4.1.3.8 to this quantum Hamiltonian reduction and proves the theorem. □

The bilinear form $(\bullet|\bullet)$ is a nondegenerate pairing between $\mathfrak{g}^e$ and $\mathfrak{g}^f$, hence coordinate ring $\mathbf{C}[S_f]_\bullet$ of the Slodowy slice is isomorphic, as commutative algebras but *not as Poisson algebras*, to the symmetric algebra $(\operatorname{Sym}\mathfrak{g}^f)_\bullet$. This isomorphism is compatible with the Kazhdan gradings on both sides and we get a graded algebra isomorphism

$$\Omega : \operatorname{gr}_\bullet \mathcal{U}(\mathfrak{g}, f) \xrightarrow{\sim} (\operatorname{Sym}\mathfrak{g}^f)_\bullet.$$

Denote by $\{x_i\}_{i=1}^c$ a linear basis of $\mathfrak{g}^f$ such that each x_i is homogeneous for the good grading, of degree $\delta(x_i)$.

Corollary 4.2.3.5 ([Pre02, Theorem 4.5])**.** *For any* $1 \leqslant i \leqslant c$*, there exists an element* X_i *belonging to* $\mathcal{U}(\mathfrak{g}, f)_{2-\delta(x_i)}$ *such that* $\Omega([X_i]_{2-\delta(x_i)}) = x_i$ *and such that the ordered products*

$$\{X_{i_1} \cdots X_{i_k} \mid k \geqslant 0,\ 1 \leqslant i_1 \leqslant \cdots \leqslant i_k \leqslant c\},$$

form a basis of $\mathcal{U}(\mathfrak{g}, f)$*.*

Remark 4.2.3.6. In [Pre02], Premet constructed a PBW basis of the finite W-algebra by using positive characteristic machinery. This corollary is also proved in [BGK08, Theorem 3.6] by using only characteristic 0.

Proof. Denote by X_i a lift of $\Omega^{-1}(x_i)$ in $\mathcal{U}(\mathfrak{g}, f)_{2-\delta(x_i)}$. Because the filtration on $\mathcal{U}(\mathfrak{g}, f)$ is nonnegative and each piece of the filtration is finite dimensional, for any $p \geqslant 0$ we have the equality

$$\dim \mathcal{U}(\mathfrak{g}, f)_p = \dim \Big(\bigoplus_{q=0}^{p} (\operatorname{Sym} \mathfrak{g}^f)_q \Big).$$

Moreover the filtration is exhaustive, so we can conclude the proof by showing that the ordered products $X_{i_1} \cdots X_{i_k}$ satisfying the additional condition $2k - (\delta(x_{i_1}) + \cdots + \delta(x_{i_k})) \leqslant p$ form a basis of $\mathcal{U}(\mathfrak{g}, f)_p$ for any $p \leqslant 0$. Because

$$\Omega([X_{i_1} \cdots X_{i_k}]_{2k-(\delta(x_{i_1})+\cdots+\delta(x_{i_k}))}) = x_{i_1} \cdots x_{i_k},$$

and because the ordered products $x_{i_1} \cdots x_{i_k}$ satisfying the same condition form a basis of $\bigoplus_{q=0}^{p} (\operatorname{Sym} \mathfrak{g}^f)_q$, we get that the ordered products $X_{i_1} \cdots X_{i_k}$ are linearly independent.

The equality of dimensions above shows that the number of ordered products $X_{i_1} \cdots X_{i_k}$ equals the dimension of $\mathcal{U}(\mathfrak{g}, f)_p$, so they form a basis. The corollary follows. □

4.3. Reduction by stages for finite W-algebras

This section is devoted to the statement and proof of reduction by stages for finite W-algebras Theorem 2 (4.3.1.3). The strategy of the proof roughly follows [GJ24]. The main point is to study the compatibility of the Kazhdan filtration induced by H_2 with the W-algebra $\mathcal{U}(\mathfrak{g}, f_1)$ (Proposition 4.3.2.1), to use the geometric reduction by stages.

4.3.1. Statement of reduction by stages

For $i = 1, 2$, let f_i be a nilpotent element and H_i be in the Cartan subalgebra $\mathfrak{h}$. Assume that the Lie algebra grading $\mathfrak{g} = \bigoplus_{\delta \in \mathbf{Z}} \mathfrak{g}_\delta^{(i)}$ defined by the adjoint action of H_i is a good grading for f_i. Since $[H_1, H_2] = 0$, the good gradings define a bigrading $\mathfrak{g} = \bigoplus_{\delta_1, \delta_2 \in \mathbf{Z}} \mathfrak{g}_{\delta_1, \delta_2}$. Set $f_0 \coloneqq f_2 - f_1$.

Assume that the conditions ($\bigstar$) holds and consider the nilpotent subalgebras $\mathfrak{n}_2$, $\mathfrak{n}_1$ and $\mathfrak{n}_0$ constructed in Section 2.3.1. Recall that the corresponding unipotent subgroups of the reductive group G satisfy the semi-direct product decomposition $N_2 = N_1 \rtimes N_0$. For $i = 1, 2$, one has by definition

$$\mathfrak{n}_i = \mathfrak{l}_i^{\perp, \omega_i} \oplus \mathfrak{g}_{\geqslant 2}^{(i)},$$

where $\mathfrak{l}_i$ is an isotropic subspace of $\mathfrak{g}_{1,1}^{(i)}$ such that there is a symplectic isomorphism $\mathfrak{g}_{1,1} \cong \mathfrak{l}_i{}^{\perp,\omega_i}/\mathfrak{l}_i$.

The group N_i acts on $\mathcal{U}(\mathfrak{g}) \otimes_{\mathbf{C}} \mathscr{W}(\mathfrak{g}_{1,1})$ with a quantum comoment map

$$\widehat{\mu}_i \coloneqq \widehat{\mu}_{\mathfrak{l}_i} : \mathfrak{n}_{\mathfrak{l}} \longrightarrow \mathcal{U}(\mathfrak{g}) \otimes_{\mathbf{C}} \mathscr{W}(\mathfrak{l}^{\perp,\omega}/\mathfrak{l})$$

defined by the formula (4.2.2.1). By Hamiltonian reduction, one gets the finite W-algebra

$$\mathcal{U}(\mathfrak{g}, f_i) = \mathcal{Q}_i{}^{N_i},$$

where $\mathcal{Q}_i \coloneqq \big(\mathcal{U}(\mathfrak{g}) \otimes_{\mathbf{C}} \mathscr{W}(\mathfrak{g}_{1,1})\big)/\widehat{I}_i$ is the quotient by the left ideal spanned by the $\widehat{\mu}_i(x)$ for x in $\mathfrak{n}_i$.

Lemma 4.3.1.1. *The following triangle commutes:*

$$\begin{tikzcd} \mathfrak{n}_2 \arrow[dr, "\widehat{\mu}_2"] & \\ & \mathcal{U}(\mathfrak{g}) \otimes_{\mathbf{C}} \mathscr{W}(\mathfrak{g}_{1,1}). \\ \mathfrak{n}_1 \arrow[uu, hook] \arrow[ur, "\widehat{\mu}_1"'] & \end{tikzcd}$$

Lemma 4.3.1.2. *The map*

$$\mathfrak{n}_0 \longrightarrow \mathcal{U}(\mathfrak{g}, f_1), \quad x \longmapsto x \otimes 1 \bmod \widehat{I}_1$$

is an embedding of Lie algebras.

Proof. The lemma follows if we prove that for all x in $\mathfrak{n}_0$, $x \otimes 1 \bmod \widehat{I}_1$ belongs to the W-algebra. We just need to check that for all y in $\mathfrak{n}_1$, $[y, x] \otimes 1$ belongs to $\widehat{I}_1$. Notice the vanishing

$$\chi_1([y, x]) = (f_1 | [y, x]) = ([x, f_1] | y) = 0,$$

because $\mathfrak{n}_0$ commutes with f_1. Hence

$$[y, x] \otimes 1 = [y, x] \otimes 1 + \chi_1([y, x])1 = \widehat{\mu}_1([y, x])$$

belongs to the ideal. $\square$

The following theorem is a new formulation of [GJ24, Main Theorem 3] under conditions ($\bigstar$). Note that $\mathfrak{l}_1$ and $\mathfrak{l}_2$ may not be Lagrangian, in contrast to [GJ24].

Theorem 4.3.1.3. *Assume the conditions ($\bigstar$). Then Proposition 2.3.1.1 holds and we can use the objects introduced just above. The following properties hold:*

1. *there is an induced action of N_0 on $\mathcal{U}(\mathfrak{g}, f_1)$ and there is an induced quantum comoment map*

$$\widehat{\mu}_0 : \mathfrak{n}_0 \longrightarrow \mathcal{U}(\mathfrak{g}, f_1), \quad x \longmapsto x \otimes 1 + \overline{\chi_0} 1 \bmod \widehat{I}_1,$$

where $\overline{\chi_0}$ denotes character obtained by restriction of χ_2 to $\mathfrak{n}_0$,

2. *the projection $\mathcal{Q}_1 \twoheadrightarrow \mathcal{Q}_2$ induces an algebra isomorphism*

$$(\mathcal{U}(\mathfrak{g}, f_1)/\widehat{I}_0)^{N_0} \cong \mathcal{U}(\mathfrak{g}, f_2),$$

where $\widehat{I}_0$ denotes the left $\mathcal{U}(\mathfrak{g}, f_1)$-ideal spanned by the $\widehat{\mu}_0(x)$ for x in $\mathfrak{n}_0$.

Proof of (1) *of Theorem 4.3.1.3.* Thanks to Lemma 4.3.1.1, Lemma 4.1.4.1 from the general theory of quantum reduction by stages can be applied to his case and we get the induced action and quantum comoment map. □

To prove (2), we want to use Proposition 4.1.4.2. There is a natural map

$$\mathcal{U}(\mathfrak{g}, f_1)/\widehat{I}_0 \longrightarrow \mathcal{Q}_2{}^{N_1}$$

provided by Proposition 4.1.4.2. If this map is an isomorphism, then the theorem follows.

To prove it is an isomorphism, we want to compare this map to the isomorphism

$$\mu_2{}^{-1}(0)//N_1 \cong \mu_0{}^{-1}(0)$$

deduced from Lemma 2.3.3.5, with the notations from Section 2.3.3. We endow our objects with relevant filtrations or gradings.

4.3.2. New quantisation of the first Slodowy slice

Recall that, for $i = 1, 2$, the group N_i acts on $\mathfrak{g}^* \times \mathfrak{g}_{1,1}$ with a moment map

$$\mu_i : \mathfrak{g}^* \times \mathfrak{g}_{1,1} \longrightarrow \mathfrak{n}_i{}^*$$

defined by the formula (2.3.3.1). Denote $\rho_2 : \mathbf{G}_{\mathrm{m}} \to \mathrm{Aut}_{\mathbf{C}}(\mathfrak{g}^*)$ the action defined by the formula (2.2.3.5) applied to the H_2-grading. There is an induced $\mathbf{G}_{\mathrm{m}}$-action on $\mathfrak{n}_i{}^*$. Moreover, $\mathfrak{g}_{1,1}$ carries the $\mathbf{G}_{\mathrm{m}}$-action by scalar multiplication. Because of Conditions (★), f_1 belongs to $\mathfrak{g}_{1,1}$ and is homogenous for both good gradings. Consider the Slodowy slice S_i associated with an H_2-homogenous $\mathfrak{sl}_2$-triple (e_i, h_i, f_i).

For $i = 1, 2$, the moment map μ_i is $\mathbf{G}_{\mathrm{m}}$-equivariant because χ_i is ρ_2-invariant. The following varieties are stabilised by the ρ_2-actions:

$$\mu_1{}^{-1}(0), \quad S_1, \quad \mu_2{}^{-1}(0), \quad S_2.$$

Denote by $\gamma_2 : \mathbf{G}_\mathrm{m} \to G_\mathrm{ad}$ the group homomorphism induced by the H_2-grading on $\mathfrak{g}$. There is a $\mathbf{G}_\mathrm{m}$-action on G defined by

$$\mathbf{G}_\mathrm{m} \xrightarrow{\gamma_2} G_\mathrm{ad} \xrightarrow{\text{conjugation}} \mathrm{Aut}(G),$$

and N_i is a $\mathbf{G}_\mathrm{m}$-stable subgroup. By the same argument as for Lemma 2.2.3.6, the isomorphism

$$\alpha_i : N_i \times S_i \longrightarrow {\mu_i}^{-1}(0), \quad (g, \xi) \longmapsto \mathrm{Ad}^*(g)\xi$$

is $\mathbf{G}_\mathrm{m}$-equivariant.

By Lemma 4.1.2.3, the Hamiltonian reduction ${\mu_1}^{-1}(0)/\!/N_1$ has an induced ρ_2-action and $\mu_0 : {\mu_1}^{-1}(0)/\!/N_1 \to {\mathfrak{n}_0}^*$ is $\mathbf{G}_\mathrm{m}$-equivariant, so $\mu^{-1}(0)$ is ρ_2-stable.

Therefore, the natural filtrations and gradings for this problem are the Kazhdan filtrations induced by the H_2-grading. We denote with a supscript "(i)" the Kazhdan filtrations and gradings associated with H_i, for $i = 1, 2$. For example $\mathcal{U}(\mathfrak{g})^{(i)}_\bullet$ is the Kazhdan filtration on the enveloping algebra. By projection or intersection, the filtration $\mathcal{U}(\mathfrak{g})^{(2)}_\bullet$ descends to:

$$\mathcal{Q}_1, \quad \mathcal{U}(\mathfrak{g}, f_1), \quad \mathcal{U}(\mathfrak{g}, f_1)/\widehat{I}_0, \quad \mathcal{Q}_2, \quad \mathcal{U}(\mathfrak{g}, f_2).$$

Proposition 4.3.2.1. *The graded Poisson algebra isomorphism*

$$\mathbf{C}[\mathfrak{g}^*]^{(2)}_\bullet \otimes_\mathbf{C} \mathbf{C}[\mathfrak{g}_{1,1}]^{(2)}_\bullet \cong \mathrm{gr}^{(2)}_\bullet \left(\mathcal{U}(\mathfrak{g}) \otimes_\mathbf{C} \mathcal{W}(\mathfrak{g}_{1,1})\right)$$

induces a graded Poisson algebra isomorphism

$$\mathbf{C}[{\mu_1}^{-1}(0)/\!/N_1]^{(2)}_\bullet \cong \mathrm{gr}^{(2)}_\bullet \mathcal{U}(\mathfrak{g}, f_1).$$

Remark 4.3.2.2. This proposition cannot be proved with Theorem 4.1.3.8 because the filtration $(\mathcal{Q}_1)^{(2)}_\bullet$ is not supported on $\mathbf{Z}_{\geqslant 0}$.

To prove the proposition, we need to introduce a grading on $\mathcal{U}(\mathfrak{g}, f_1)$ and study its relation with the two Kazhdan filtrations. Set $H_0 := H_2 - H_1$, its adjoint action induces an action $\gamma_0 : \mathbf{G}_\mathrm{m} \to \mathrm{Aut}_\mathrm{Alg}(\mathcal{U}(\mathfrak{g}))$ by algebra automorphism. Denote by $\mathcal{U}(\mathfrak{g})[\delta]$ the weight space corresponding to δ in $\mathbf{Z}$. Note the inclusion $\mathfrak{g}_{\delta_1,\delta_2} \subseteq \mathcal{U}(\mathfrak{g})[\delta_2 - \delta_1]$ for all δ_1, δ_2 in $\mathbf{Z}$. By considering the trivial action on the Weyl algebra, it extends to an action

$$\gamma_0 : \mathbf{G}_\mathrm{m} \to \mathrm{Aut}_\mathrm{Alg}(\mathcal{U}(\mathfrak{g}) \otimes_\mathbf{C} \mathcal{W}(\mathfrak{g}_{1,1})).$$

Lemma 4.3.2.3. *The ideal $\widehat{I}_1$ is $\mathbf{G}_\mathrm{m}$-stable so the quotient $\mathcal{Q}_1$ has an induced γ_0-action. This action stabilises $\mathcal{U}(\mathfrak{g}, f_1)$, so we get an induced action*

$$\gamma_0 : \mathbf{G}_\mathrm{m} \to \mathrm{Aut}_\mathrm{Alg}(\mathcal{U}(\mathfrak{g}, f_1))$$

and the weight spaces correspond to the adjoint action of $H_0 \otimes 1 \bmod \widehat{I}_1$, that belongs to $\mathcal{U}(\mathfrak{g}, f_1)$.

Proof. It is enough to prove that the element $H_0 \otimes 1 \bmod \widehat{I_1}$, denoted by H_0, belongs to $\mathcal{U}(\mathfrak{g}, f_1)$. For all x in $\mathfrak{n}_0$, we have the vanishing

$$\chi_1([y, H_0]) = (f_1|[y, H_0]) = (y|[H_0, f_1]) = 0$$

because $[H_0, f_1] = 0$. Hence, since $\mathfrak{n}_1$ is H_0-stable,

$$[y, H_0] = [y, H_0] \otimes 1 + \chi_1([y, H_0])$$

belongs to $\widehat{I_1}$ and H_0 belongs to $\mathcal{U}(\mathfrak{g}, f_1)$. $\square$

Because H_0 commutes with H_1 and H_2, the action γ_0 is compatible with the Kazhdan filtrations coming from H_1 and H_2 on $\mathcal{U}(\mathfrak{g})$, $\mathcal{Q}_1$ and $\mathcal{U}(\mathfrak{g}, f_1)$. Denote with postfix "$[\delta]$" the corresponding weight spaces.

Lemma 4.3.2.4. *For p in $\mathbf{Z}$, the subspace $\mathcal{U}(\mathfrak{g})_p^{(1)}$ is H_0-stable and the Kazhdan filtrations are related by the relation:*

$$\mathcal{U}(\mathfrak{g})_p^{(2)} = \bigoplus_{\delta \in \mathbf{Z}} \mathcal{U}(\mathfrak{g})_{p+\delta}^{(1)}[\delta].$$

The same relation holds for $\mathcal{Q}_1$ and $\mathcal{U}(\mathfrak{g}, f_1)$.

We use the same kind of arguments as in the proof of Lemma 4.2.1.1.

Proof. Because $[H_1, H_2] = 0$, we can use the Poincaré–Birkhoff–Witt Theorem to check that the subspace $\mathcal{U}(\mathfrak{g})_p^{(2)}$ is exactly spanned by the products $x_1 \cdots x_n$ where n is a nonnegative integer and x_i belongs to $\mathfrak{g}_{\delta_1(x_i), \delta_2(x_i)}$ for $\delta_1(x_i), \delta_2(x_i)$ in $\mathbf{Z}$ such that

$$2n - (\delta_2(x_1) + \cdots + \delta_2(x_n)) \leqslant p.$$

Set $\delta_0(x_i) \coloneqq \delta_2(x_i) - \delta_1(x_i)$. The previous inequality is equivalent to

$$q - (\delta_0(x_1) + \cdots + \delta_0(x_n)) \leqslant p, \quad \text{where} \quad q \coloneqq 2n + (\delta_1(x_1) + \cdots + \delta_1(x_n)).$$

For the reverse, we need to check that for δ, q satisfying $q - \delta \leqslant p$, if F belongs to $\mathcal{U}(\mathfrak{g})_q^{(1)}$ and if $[H_0, F] = \delta F$, then F belongs to $\mathcal{U}(\mathfrak{g})_p^{(2)}$. To do so, we use the PBW basis to prove that F is sum of products $x_1 \cdots x_n$ such that

$$\delta_0(x_1) + \cdots + \delta_0(x_n) = \delta \quad \text{and} \quad 2n - (\delta_1(x_1) + \cdots + \delta_1(x_n)) \leqslant q.$$

By taking the difference, we get

$$2n - (\delta_2(x_1) + \cdots + \delta_2(x_n)) \leqslant p,$$

so F is indeed a sum of terms in $\mathcal{U}(\mathfrak{g})_p^{(2)}$. $\square$

Lemma 4.3.2.5. *Let $V_\bullet$ be a filtered vector space equipped with a filtration-preserving $\mathbf{G}_\mathrm{m}$-module structure. Then $\operatorname{gr}_\bullet V$ is equipped with $\mathbf{G}_\mathrm{m}$-module structure that preserves the grading.*

Denote by $V_p[\delta]$ and $(\operatorname{gr}_p V)[\delta]$ the weight spaces of the $\mathbf{G}_\mathrm{m}$-actions for δ, p in $\mathbf{Z}$. There is a linear isomorphism

$$\operatorname{gr}_p(V[\delta]) \cong (\operatorname{gr}_p V)[\delta]$$

induced by the inclusion $V[\delta] \subseteq V$.

Proof. The first part of the statement is a consequence of Lemma 4.1.2.4 applied to $\mathbf{G}_\mathrm{m}$, seen as a trivially graded group so that V is a filtered $\mathbf{G}_\mathrm{m}$-module.

The filtration on $V[\delta]$ is defined by

$$V[\delta]_p \coloneqq V[\delta] \cap V_p = V_p[\delta].$$

The inclusion $V[\delta]_\bullet \subseteq V_\bullet$ respects the filtrations so we get a graded inclusion $\operatorname{gr}_\bullet(V[\delta]) \subseteq (\operatorname{gr}_\bullet V)[\delta]$. But $\operatorname{gr}_p V$ decomposes in two possible ways:

$$\bigoplus_{\delta \in \mathbf{Z}} \operatorname{gr}_p(V[\delta]) = \operatorname{gr}_p V = \bigoplus_{\delta \in \mathbf{Z}} (\operatorname{gr}_p V)[\delta],$$

hence the inclusion $\operatorname{gr}_p(V[\delta]) \subseteq (\operatorname{gr}_p V)[\delta]$ is an equality. $\square$

Proof of Proposition 4.3.2.1. Recall that the graded Poisson algebra isomorphism

$$\mathbf{C}[\mathfrak{g}^*]^{(2)}_\bullet \otimes_\mathbf{C} \mathbf{C}[\mathfrak{g}_{1,1}]^{(2)}_\bullet \cong \operatorname{gr}^{(2)}_\bullet \left(\mathcal{U}(\mathfrak{g}) \otimes_\mathbf{C} \mathcal{W}(\mathfrak{g}_{1,1})\right)$$

is defined by sending x in $\mathfrak{g}^{(2)}_\delta$ to $[x]^{(2)}_{2-\delta}$ for δ in $\mathbf{Z}$. Because $\operatorname{gr}^{(2)} \widehat{\mu}_1$ coincides with μ_1, the graded isomorphisms identifies the graded ideals $(I_1)^{(2)}_\bullet$ and $\operatorname{gr}^{(2)}_\bullet I_1$, so we get the induced graded isomorphism

$$\mathbf{C}[{\mu_1}^{-1}(0)]^{(2)}_\bullet \cong \operatorname{gr}^{(2)}_\bullet \mathcal{Q}_1.$$

Notice that this isomorphism is compatible with the graded N_1-actions on both sides, hence we get a graded Poisson isomorphism

$$\mathbf{C}[{\mu_1}^{-1}(0)//N_1]^{(2)}_\bullet \cong (\operatorname{gr}^{(2)}_\bullet \mathcal{Q}_1)^{N_1},$$

where the right-hand side has a Poisson bracket by Hamiltonian reduction. Moreover there is a tautological Poisson embedding

$$\operatorname{gr}^{(2)}_\bullet \mathcal{U}(\mathfrak{g}, f_1) \longrightarrow (\operatorname{gr}^{(2)}_\bullet \mathcal{Q}_1)^{N_1}.$$

To conclude, it is enough to show that this embedding is an isomorphism.

By using Lemmas 4.3.2.4 and 4.3.2.5, we can write the following decompositions for p in $\mathbf{Z}$:

$$\begin{aligned}
\operatorname{gr}^{(2)}_p \mathcal{U}(\mathfrak{g}, f_1) &= \bigoplus_{\delta\in\mathbf{Z}} \operatorname{gr}^{(1)}_{p+\delta} \mathcal{U}(\mathfrak{g}, f_1)[\delta], \\
(\operatorname{gr}^{(2)}_p \mathcal{Q}_1)^{N_1} &= \Big(\bigoplus_{\delta\in\mathbf{Z}} \operatorname{gr}^{(1)}_{p+\delta} \mathcal{Q}_1[\delta]\Big)^{N_1} = \bigoplus_{\delta\in\mathbf{Z}} (\operatorname{gr}^{(1)}_{p+\delta} \mathcal{Q}_1)^{N_1}[\delta],
\end{aligned}$$

where the last equality follows from Lemma 4.1.2.3 applied to $(\operatorname{gr}^{(2)}_p \mathcal{Q}_1)[\bullet]$ seen as a graded N_1-module. The isomorphism $\operatorname{gr}^{(1)}_q \mathcal{U}(\mathfrak{g}, f_1) \cong (\operatorname{gr}^{(1)}_q \mathcal{Q}_1)^{N_1}$ given by Theorem 4.2.3.1 restricts to a linear isomorphism

$$\operatorname{gr}^{(1)}_q \mathcal{U}(\mathfrak{g}, f_1)[\delta] \cong (\operatorname{gr}^{(1)}_q \mathcal{Q}_1)^{N_1}[\delta].$$

Hence the isomorphism

$$\operatorname{gr}^{(2)}_p \mathcal{U}(\mathfrak{g}, f_1) \cong (\operatorname{gr}^{(2)}_p \mathcal{Q}_1)^{N_1}$$

follows from the fact that their decompositions coincide. □

Remark 4.3.2.6. The above proof uses the same main ideas as [GJ24, Proposition 3.3.1], but we noticed some mistake in the proof. In Claim 3.3.5, it is stated that the isomorphism $\operatorname{gr}^{(1)} \mathcal{Q}_1 \cong \operatorname{gr}^{(2)} \mathcal{Q}_1$ induces an embedding

$$\operatorname{gr}^{(1)} \mathcal{U}(\mathfrak{g}, f_1) \longhookrightarrow \operatorname{gr}^{(2)} \mathcal{U}(\mathfrak{g}, f_1)$$

by taking the N_1-invariants. In fact one can only deduce an embedding

$$\operatorname{gr}^{(1)} \mathcal{U}(\mathfrak{g}, f_1) \longhookrightarrow (\operatorname{gr}^{(2)} \mathcal{Q}_1)^{N_1}.$$

4.3.3. Proof of reduction by stages

Recall that to finish the proof of reduction by stages, we want to compute the graded homomorphism corresponding to

$$\mathcal{U}(\mathfrak{g}, f_1)/\widehat{I}_0 \longrightarrow {\mathcal{Q}_2}^{N_1}$$

for the Kazhdan filtration associated with H_2. We start by compute the graded objects corresponding to the domain and codomain.

Lemma 4.3.3.1. *There is a natural graded algebra isomorphism*

$$\mathbf{C}[{\mu_0}^{-1}(0)]^{(2)}_\bullet \cong \operatorname{gr}^{(2)}_\bullet \big(\mathcal{U}(\mathfrak{g}, f_1)/\widehat{I}_0\big).$$

Proof. The identification ${\mu_0}^* = \operatorname{gr}^{(2)} \widehat{\mu}_0$ is clear by comparing their formulae. The isomorphism is a consequence of the exact sequence isomorphism given by Lemma 4.1.3.7 applied to this quantum comoment map. □

Lemma 4.3.3.2. *There are natural graded algebra isomorphisms*

$$\operatorname{gr}^{(2)}_{\bullet}({\mathcal{Q}_2}^{N_1}) \cong (\operatorname{gr}^{(2)}_{\bullet} \mathcal{Q}_2)^{N_1} \cong \mathbf{C}[{\mu_2}^{-1}(0)//N_1]^{(2)}_{\bullet}.$$

Proof. Because $(\mathcal{Q}_2)^{(2)}_{\bullet}$ is nonnegatively filtered, the first isomorphism is provided by Proposition 4.1.2.5. The second isomorphisms follows from the fact that the isomorphism

$$\mathbf{C}[{\mu_2}^{-1}(0)]^{(2)}_{\bullet} \cong \operatorname{gr}^{(2)}_{\bullet} \mathcal{Q}_2$$

given by Lemma 4.2.3.3 is N_1-equivariant and compatible with the grading on N_1. □

Before going to the end of the proof, we check that the filtrations are nonnegatively supported on both sides. It is clear for the codomain because it is true for $\mathcal{Q}_2$.

Lemma 4.3.3.3. *The ideal $\widehat{I}_0$ contains $\mathcal{U}(\mathfrak{g}, f_1)^{(2)}_{-1}$. It follows that $\mathcal{U}(\mathfrak{g}, f_1)/\widehat{I}_0$ is nonnegatively filtered: $(\mathcal{U}(\mathfrak{g}, f_1)/\widehat{I}_0)^{(2)}_{-1} = 0$.*

Proof. Let $\{x_i\}_{i=1}^{c}$ be an ordered basis of $\mathfrak{g}^{f_1}$ that is homogeneous for the bigrading, in particular x_i belongs to $\mathfrak{g}_{\delta_1(x_i),\delta_2(x_i)}$ for some $\delta_1(x_i), \delta_2(x_i)$ in $\mathbf{Z}$ and x_i belongs to $\mathbf{C}[S_1]^{(1)}_{2-\delta_1(x_i)}[\delta_2(x_i) - \delta_1(x_i)]$. Recall the isomorphism

$$\operatorname{gr}^{(1)}_p(\mathcal{U}(\mathfrak{g}, f_1)[\delta]) \cong \mathbf{C}[S_1]^{(1)}_p[\delta]$$

for any p, δ in $\mathbf{Z}$. Denote by X_i an element of $\mathcal{U}(\mathfrak{g}, f_1)^{(1)}_{2-\delta_1(x_i)}[\delta_2(x_i) - \delta_1(x_i)]$ that lifts x_i. In particular X_i belongs to $\mathcal{U}(\mathfrak{g}, f_1)^{(2)}_{2-\delta_2(x_i)}$. By Corollary 4.2.3.5, the ordered products

$$\{X_{i_1} \cdots X_{i_k} \mid k \geqslant 0,\ 1 \leqslant i_1 \leqslant \cdots \leqslant i_k \leqslant c\},$$

form a basis of $\mathcal{U}(\mathfrak{g}, f_1)$.

In the proof of the corollary, it is proved that the subfamily

$$\{X_{i_1} \cdots X_{i_k} \mid 2k - (\delta_1(x_{i_1}) + \cdots + \delta_1(x_{i_k})) \leqslant q)\}$$

is a basis of $\mathcal{U}(\mathfrak{g}, f_1)^{(1)}_q$ for q in $\mathbf{Z}$. By using Lemma 4.3.2.4, we see that the subfamily

$$\{X_{i_1} \cdots X_{i_k} \mid 2k - (\delta_2(x_{i_1}) + \cdots + \delta_2(x_{i_k})) \leqslant p)\}$$

is a basis of $\mathcal{U}(\mathfrak{g}, f_1)^{(2)}_p$ for p in $\mathbf{Z}$.

Consider a decomposition

$$\mathfrak{g} = \mathfrak{n}_1 \oplus \mathfrak{n}_0 \oplus \mathfrak{c}_0 \oplus \mathfrak{c}_1,$$

that is compatible with the bigrading and satisfying

$$\mathfrak{g}^{f_1} = \mathfrak{n}_0 \oplus \mathfrak{c}_0.$$

Because $\mathfrak{n}_2 = \mathfrak{n}_1 \oplus \mathfrak{n}_0$ is contained in $\mathfrak{g}^{(2)}_{\geqslant 1}$, then $\mathfrak{c}_0 \oplus \mathfrak{c}_1$ is contained $\mathfrak{g}^{(2)}_{\leqslant 0}$.

By taking a basis $\{x_i\}_{i=1}^{c}$ adapted to the decomposition $\mathfrak{g}^{f_1} = \mathfrak{n}_0 \oplus \mathfrak{c}_0$, we see that $\mathcal{U}(\mathfrak{g}, f_1)^{(2)}_{-1}$ is contained in the left-ideal spanned by the $x \otimes 1 \bmod \widehat{I}_1$, where x is taken in $\mathfrak{n}_0 \cap \mathfrak{g}^{(2)}_{\geqslant 3}$, and this ideal is contained in the left ideal spanned by $\widehat{\mu}_0(\mathfrak{n}_0)$. $\square$

We have all the tools to conclude the proof of the main theorem.

Proof of (2) *of Theorem 4.3.1.3.* The filtered homomorphism

$$\mathcal{U}(\mathfrak{g}, f_1)/\widehat{I}_0 \longrightarrow \mathcal{Q}_2{}^{N_1} \tag{4.3.3.4}$$

has for domain and codomain objects equipped with exhaustive and nonnegatively supported filtrations, according to Lemma 4.3.3.3. Then by a standard fact (recalled in [KPW22, Lemma 6.8]), if the associated graded homomorphism is bijective then so is the filtered one.

But as a direct consequence of Lemmas 4.3.3.1 and 4.3.3.2, the graded map associated with (4.3.3.4) appears to be the comorphism corresponding to the variety isomorphism

$$\mu_2{}^{-1}(0)//N_1 \cong \mu_0{}^{-1}(0).$$

This concludes the proof. $\square$

4.4. Hamiltonian reduction and modules

In this section, following [GG06], we recall the existence of a pair of adjoint functors between the categories of modules on an associative unital algebra and its quantum Hamiltonian reduction (Proposition 4.4.1.2). We prove a generalisation of the famous Skryabin equivalence (Theorem 4.4.2.2) and we deduce Theorem 3 as a particular case.

4.4.1. Hamiltonian reduction functor

Let N be a *connected* linear algebraic group. Let $\mathcal{A}$ be an associative unital algebra with an a lgebraic action of a linear algebraic group N by algebra automorphisms. Assume the existence of $\widehat{\mu} : \mathfrak{n} \to \mathcal{A}$, an N-equivariant comoment map. Denote by $\mathbf{C}|0\rangle$ the trivial $\mathfrak{n}$-module and consider the induced $\mathcal{A}$-module

$$\mathcal{Q} := \mathcal{A} \otimes_{\mathcal{U}(\mathfrak{n})} \mathbf{C}|0\rangle,$$

where $\mathcal{A}$ is a right $\mathcal{U}(\mathfrak{n})$-module through the quantum comoment map. Note the $\mathcal{A}$-equivariant isomorphism

$$\mathcal{A}/\widehat{I} \xrightarrow{\sim} \mathcal{Q}, \quad (a \bmod \widehat{I}) \longmapsto a \otimes |0\rangle,$$

where $\widehat{I}$ is the left ideal spanned by the $\widehat{\mu}(x)$ for x in $\mathfrak{n}$. In particular, is $\mathcal{Q}^N$ the quantum Hamiltonian reduction with respect to $\widehat{\mu}$.

Denote by $\mathrm{Hom}_{\mathcal{A}}(\bullet, \bullet)$ the bifunctor giving the space $\mathcal{A}$-equivariant homomorphisms between to left $\mathcal{A}$-modules. For any left $\mathcal{A}$-module V, the comoment map $\widehat{\mu} : \mathfrak{n} \to \mathcal{A}$ equips V with a structure of $\mathfrak{n}$-module.

Lemma 4.4.1.1 ([GG06, Section 7])**.** *Recall that we assumed that N is connected.*

1. *For any left $\mathcal{A}$-module V, there is a $\mathbf{C}$-linear isomorphism*
$$\mathrm{Hom}_{\mathcal{A}}(\mathcal{Q}, V) \xrightarrow{\sim} V^{\mathfrak{n}}, \quad \Phi \longmapsto \Phi(1 \otimes |0\rangle)$$
that is natural in V.

2. *In particular, the following algebra isomorphism holds*
$$\mathrm{End}_{\mathcal{A}}(\mathcal{Q})^{\mathrm{op}} \xrightarrow{\sim} \mathcal{Q}^N, \quad \Phi \longmapsto \Phi(1 \otimes |0\rangle),$$
where the codomain is the algebra of $\mathcal{A}$-linear endomorphism of $\mathcal{Q}$ with the opposite multiplication.

Proof. Because $\mathcal{Q}$ is spanned by $1 \otimes |0\rangle$ as $\mathcal{A}$-module, any $\mathcal{A}$-module homomorphism $\Phi : \mathcal{Q} \to V$ is entirely determined by $\Phi(1 \otimes |0\rangle)$. For any x in $\mathfrak{n}$, one has
$$\widehat{\mu}(x) \cdot \Phi(1 \otimes |0\rangle) = \Phi(\widehat{\mu}(x) \otimes |0\rangle) = \Phi\big(1 \otimes (x \cdot |0\rangle)\big) = 0,$$
so $\Phi(1 \otimes |0\rangle)$ belongs to $V^{\mathfrak{n}}$. In the other direction, for any v in $V^{\mathfrak{n}}$, we can define a map
$$\widetilde{\Phi} : \mathcal{U}(\mathfrak{g}) \longrightarrow V, \quad F \longmapsto F \cdot v,$$
and, for any x in $\mathfrak{n}$, one has
$$\widetilde{\Phi}(\widehat{\mu}(x)) = \widehat{\mu}(x) \cdot v = 0.$$
So, after taking quotient by $\widehat{I}$, there is an induced homomorphism $\Phi : \mathcal{Q} \to V$ of $\mathcal{A}$-modules and (1) is proved.

Because of the quantum comoment map and the connectedness of N, the N-invariants in $\mathcal{Q}$ are the same as the $\mathrm{ad}(\widehat{\mu}(\mathfrak{n}))$-invariants. But for any x in $\mathfrak{n}$ and any F in $\mathcal{U}(\mathfrak{g})$:
$$\begin{aligned}[\widehat{\mu}(x), F] \otimes |0\rangle =& (\widehat{\mu}(x)F) \otimes |0\rangle - (F\widehat{\mu}(x)) \otimes |0\rangle \\ =& (\widehat{\mu}(x)F) \otimes |0\rangle - F \otimes (x \cdot |0\rangle) \\ =& (\widehat{\mu}(x)F) \otimes |0\rangle,\end{aligned}$$
so the N-invariants in $\mathcal{Q}$ are the same as the $\widehat{\mu}(\mathfrak{n})$-invariants (for the left-multiplication), summarized in a nutshell:
$$\mathcal{Q}^N = \mathcal{Q}^{\mathrm{ad}(\widehat{\mu}(\mathfrak{n}))} = \mathcal{Q}^{\widehat{\mu}(\mathfrak{n})} =: \mathcal{Q}^{\mathfrak{n}}.$$
(2) follows from (1) after checking that the isomorphism is compatible with the multiplicative structures. $\square$

Note that $\mathrm{End}_{\mathcal{A}}(\mathcal{Q})^{\mathrm{op}} \cong \mathcal{Q}^N$ acts from the right on $\mathcal{Q}$ and from the left on $\mathrm{Hom}_{\mathcal{A}}(\mathcal{Q}, V) \cong V^{\mathfrak{n}}$ for any left $\mathcal{A}$-module V. Denote by $\mathcal{A}$-Mod the category of left $\mathcal{A}$-modules.

Proposition 4.4.1.2 ([GG06, Section 7])**.** *The functors*

$$\mathrm{Inv} : V \longmapsto V^{\mathfrak{n}} \quad \textit{and} \quad \mathrm{Ind} : W \longmapsto \mathcal{Q} \otimes_{\mathcal{Q}^N} W$$

form an adjunction

$$\begin{tikzcd} \mathcal{A}\text{-}\mathsf{Mod} \arrow[rr, bend left, "\mathrm{Inv}"] \arrow[rr, phantom, "\perp"] & & \mathcal{Q}^N\text{-}\mathsf{Mod}. \arrow[ll, bend left, "\mathrm{Ind}"] \end{tikzcd}$$

Proof. Because of Lemma 4.4.1.1, this adjunction follows from the well-known adjunction isomorphism

$$\mathrm{Hom}_{\mathcal{A}}(\mathcal{Q} \otimes_{\mathcal{Q}^N} W, V) \cong \mathrm{Hom}_{\mathcal{Q}^N}(W, \mathrm{Hom}_{\mathcal{A}}(\mathcal{Q}, V))$$

that is natural in V in $\mathcal{A}$-Mod and W in $\mathcal{Q}^N$-Mod. □

4.4.2. Generalisation of the Skryabin equivalence

In this section we assume that N is unipotent and that $\mathfrak{n} = \bigoplus_{\delta \in \mathbf{Z}_{<0}}$ is negatively graded as a Lie algebra. Assume that $\mathcal{A}_\bullet$ is almost commutative of degree $-k$ for $k \geqslant 1$ an integer, that the comoment map $\widehat{\mu} : \mathfrak{n} \to \mathcal{A}$ is of degree $+k$ so that $\mathcal{A}$ is a filtered N-module.

A left $\mathcal{A}$-module V is called a *Whittaker module* with respect to the quantum comoment map $\widehat{\mu}$ if for all x in $\mathfrak{n}$, $\widehat{\mu}(x)$ acts locally nilpotently on V, that is to say if for all v in V, there is some positive integer n such that

$$\widehat{\mu}(x)^n \cdot v = 0.$$

Denote by $\mathsf{Wh}(\mathcal{A}, \mathfrak{n}, \widehat{\mu})$ the category of Whittaker modules.

Lemma 4.4.2.1. *Assume that the filtration on $\mathcal{Q}_\bullet$ is nonnegative. Then for any left $\mathcal{Q}^N$-module W, $\mathrm{Ind}(W) = \mathcal{Q} \otimes_{\mathcal{Q}^N} W$ is a Whittaker module.*

Proof. Take x in $\mathfrak{n}_\delta$ and F in $\mathcal{Q}_p$ for $\delta < 0$ and $p \geqslant 0$ two integers, and w in W. According to a computation done in the proof of Lemma 4.4.1.1, for any integer n:

$$\widehat{\mu}(x)^n \cdot (F \otimes w) = \mathrm{ad}(\widehat{\mu}(x))^n F \otimes w.$$

Because $\widehat{\mu}$ is of degree $+k$, $\mathrm{ad}(\widehat{\mu}(x))^n F$ belongs to $\mathcal{A}_{p+n\delta}$ that is zero for n big enough so that $p + n\delta < 0$: this is the Whittaker module condition. □

The following theorem is a natural generalisation of the *Skryabin equivalence of categories* [Skr02].

Theorem 4.4.2.2. *Let N be a unipotent algebraic group such that its Lie algebra $\mathfrak{n} = \bigoplus_{\delta \in \mathbf{Z}_{<0}} \mathfrak{n}_\delta$ is negatively graded. Let $\mathcal{A}_\bullet$ be an almost commutative algebra of degree $-k$ with an action of N by algebra automorphisms and let $\widehat{\mu} : \mathfrak{m} \to \mathcal{A}$ be an N-equivariant quantum comoment map of degree k. Define the affine Poisson scheme $X \coloneqq \operatorname{Spec}(\operatorname{gr}_\bullet A)$ and the moment map $\mu : X \to \mathfrak{m}^*$ such that $\operatorname{gr} \widehat{\mu} = \mu^*$. Make the following assumptions:*

1. *the induced filtration on the quotient $\mathcal{Q} \coloneqq \mathcal{A}/\widehat{I}$ is nonnegative,*
2. *there is a $\mathbf{G}_{\mathrm{m}}$-stable closed subscheme S of $\mu^{-1}(0)$ such that the action map*
$$\alpha : N \times S \longrightarrow \mu^{-1}(0), \quad (g, x) \longmapsto g \cdot x$$
is a $\mathbf{G}_{\mathrm{m}}$-equivariant isomorphism.

Then there is an equivalence of categories

$$\begin{tikzcd} \mathsf{Wh}(\mathcal{A}, \mathfrak{n}, \widehat{\mu}) \arrow[r, bend left, "\mathrm{Inv}"] \arrow[r, phantom, "\simeq"] & \mathcal{Q}^N\text{-}\mathsf{Mod} \arrow[l, bend left, "\mathrm{Ind}"] \end{tikzcd}$$

induced by the adjunction of Proposition 4.4.1.2.

To prove this theorem, we generalise the arguments of [GG02, Section 6]. Consider the linear map

$$\begin{aligned} \Theta_1 : W &\longrightarrow \operatorname{Inv} \circ \operatorname{Ind}(W) = (\mathcal{Q} \otimes_{\mathcal{Q}^N} W)^{\mathfrak{n}}, \\ w &\longmapsto 1 \otimes |0\rangle \otimes w \end{aligned}$$

and

$$\begin{aligned} \Theta_2 : \operatorname{Ind} \circ \operatorname{Inv}(V) = \mathcal{Q} \otimes_{\mathcal{Q}^N} V^{\mathfrak{n}} &\longrightarrow V, \\ (F \otimes |0\rangle) \otimes v &\longmapsto F \cdot v, \end{aligned}$$

that are natural in V in $\mathsf{Wh}(\mathcal{A}, \mathfrak{n}, \widehat{\mu})$ and W in $\mathcal{Q}$-Mod. If we prove that they are isomorphisms, then Theorem 4.4.2.2 is proved. Recall that by Theorem 4.1.3.8, one has the graded Poisson isomorphism

$$\mathbf{C}[\mu^{-1}(0)//N] \cong \operatorname{gr}_\bullet(\mathcal{Q}^N).$$

Proof of the bijectivity of Θ_1. To prove that Θ_1 is an isomorphism, we introduce filtrations on the objects and prove that $\operatorname{gr} \Theta_1$ is an isomorphism.

Assume first that W is finitely generated, $W = \mathcal{Q}^N \cdot W_0$ where W_0 is a finite dimensional subspace of W. For $p \geqslant 0$, set

$$W_p \coloneqq \mathcal{Q}^N_p \cdot W_0,$$

where the filtration on $\mathcal{Q}^N_\bullet$ is defined by the filtration on $\mathcal{A}_\bullet$.

The graded module $\mathrm{gr}_\bullet \mathcal{Q} \cong \mathbf{C}[\mu^{-1}(0)]_\bullet$ is a free over $\mathrm{gr}_\bullet(\mathcal{Q}^N) \cong \mathbf{C}[S]_\bullet$ because of the $\mathbf{G}_\mathrm{m}$-equivariant isomorphism $N \times S \cong \mu^{-1}(0)$. By [NO06, Lemma I.5.1.3], since all filtrations are nonnegative, $\mathcal{Q}_\bullet$ is is a free filtered right $\mathcal{Q}^N_\bullet$-module.

The filtration on $W_\bullet$ is nonnegative and, by [NO06, Lemma 8.2], the natural graded homomorphism

$$(\mathrm{gr}_\bullet \mathcal{Q}) \otimes_{\mathrm{gr}_\bullet \mathcal{Q}^N} (\mathrm{gr}_\bullet W) \longrightarrow \mathrm{gr}_\bullet(\mathcal{Q} \otimes_{\mathcal{Q}^N} W), \quad [F] \otimes [w] \longmapsto [F \otimes w]$$

is an isomorphism because $\mathcal{Q}_\bullet$ is is a free filtered right $\mathcal{Q}^N_\bullet$-module.

From this isomorphism and from the isomorphism $N \times S \cong \mu^{-1}(0)$, we deduce the $\mathfrak{n}$-equivariant isomorphism

$$\mathrm{gr}_\bullet(\mathcal{Q} \otimes_{\mathcal{Q}^N} W) \cong \mathbf{C}[N]_\bullet \otimes_{\mathbf{C}} (\mathrm{gr}_\bullet W).$$

We can apply Proposition 4.1.2.5 and get the natural isomorphism

$$\mathrm{gr}_\bullet \left((\mathcal{Q} \otimes_{\mathcal{Q}^N} W)^\mathfrak{n}\right) \cong \left(\mathrm{gr}_\bullet(\mathcal{Q} \otimes_{\mathcal{Q}^N} W)\right)^\mathfrak{n} \cong \mathrm{gr}_\bullet W.$$

Up to these isomorphism, we see that $\mathrm{gr}_\bullet \Theta_1$ coincides with the identity on $\mathrm{gr}_\bullet W$. Because the filtrations are nonnegative and exhaustive, we deduce that Θ_1 is an isomorphism by [KPW22, Lemma 6.8].

If W is not finitely generated, it is a filtered limit of its finitely generated submodules. Because Θ_1 is defined by taking a tensor product and invariants, it commutes with filtered limits and it is still an isomorphism □

To prove the bijectivity of Θ_2, we will need the following lemma.

Lemma 4.4.2.3. *For any Whittaker module V, if $V^\mathfrak{n} = 0$ then $V = 0$.*

Proof. Let v be in V and x be in $\mathfrak{n}$. There is an integer n such that $\widehat{\mu}(x)^n \cdot v = 0$. If $n = 0$, then $v = 0$. If $n > 0$ then

$$\widehat{\mu}(x)^n \cdot v = \widehat{\mu}(x) \cdot (\widehat{\mu}(x)^{n-1} \cdot x) = 0,$$

hence $\widehat{\mu}(x)^{n-1} \cdot x$ belongs to $V^\mathfrak{n} = 0$. By decreasing induction on n, we get that $v = 0$, hence $V = 0$. □

We can finish the proof of Theorem 4.4.2.2.

Proof of the bijectivity of Θ_2. Define

$$V' \coloneqq \operatorname{Ker} \Theta_2 \quad \text{and} \quad V'' \coloneqq V/\Theta_2(\mathcal{Q} \otimes_{\mathcal{Q}^N} W^\mathfrak{n})$$

the kernel and cokernel of Θ_2. Note that Θ_2 is $\mathfrak{n}$-equivariant.

The isomorphism Θ_1 applied to $W \coloneqq V^\mathfrak{n}$ gives the quality

$$(\mathcal{Q} \otimes_{\mathcal{Q}^N} V^\mathfrak{n})^\mathfrak{n} = \{(1 \otimes |0\rangle) \otimes v \mid v \in V^\mathfrak{n}\}.$$

So we get

$$\begin{aligned}(V')^{\mathfrak{n}} &= V' \cap \{(1 \otimes |0\rangle) \otimes v \mid v \in V^{\mathfrak{n}}\}\\ &= \{(1 \otimes |0\rangle) \otimes v \mid v \in V^{\mathfrak{n}},\ \Theta_2((1 \otimes |0\rangle) \otimes v) = 0\}\\ &= \{(1 \otimes |0\rangle) \otimes v \mid v \in V^{\mathfrak{n}},\ v = 0\}\\ &= 0,\end{aligned}$$

so, by Lemma 4.4.2.3, V' is zero and Θ_2 is injective.

Consider the exact sequence

$$0 \longrightarrow \mathcal{Q} \otimes_{\mathcal{Q}^N} V^{\mathfrak{n}} \xrightarrow{\Theta_2} V \longrightarrow V'' \longrightarrow 0,$$

it is a sequence of $\mathfrak{n}$-equivariant homomorphisms. We can apply the Lie algebra cohomology functor $\mathrm{H}^\bullet(\mathfrak{n}, \bullet)$, we get a long exact sequence. But by the first part of the proof, we can apply Proposition 4.1.2.5 to any finitely generated $\mathcal{Q}^N$-submodule W of $V^{\mathfrak{n}}$ and we get the vanishing

$$\mathrm{H}^1(\mathfrak{n}, \mathcal{Q} \otimes_{\mathcal{Q}^N} V^{\mathfrak{n}}) = 0,$$

because the cohomology commutes with filtered limits.

Then we have an exact sequence

$$0 \longrightarrow (\mathcal{Q} \otimes_{\mathcal{Q}^N} V^{\mathfrak{n}})^{\mathfrak{n}} \longrightarrow V^{\mathfrak{n}} \longrightarrow (V'')^{\mathfrak{n}} \longrightarrow 0.$$

The map

$$(\mathcal{Q} \otimes_{\mathcal{Q}^N} V^{\mathfrak{n}})^{\mathfrak{n}} \longrightarrow V^{\mathfrak{n}}, \quad (1 \otimes |0\rangle) \otimes v \longmapsto v$$

is clearly surjective so $(V'')^{\mathfrak{n}}$ is zero, and by Lemma 4.4.2.3, V'' is zero. The bijectivity of Θ_2 is proved. □

4.4.3. Application to Slodowy slices and reduction by stages

Let $\mathfrak{g}$ be a simple finite-dimensional complex Lie algebra, $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and G be a connected algebraic group whose Lie algebra is $\mathfrak{g}$. Denote by $(\bullet|\bullet)$ the normalised non-degenerate symmetric bilinear form on $\mathfrak{g}$. Let f a nilpotent element of $\mathfrak{g}$ and $\mathfrak{g} = \bigoplus_{\delta \in \mathbf{Z}} \mathfrak{g}_\delta$ be a good grading for f defined by the adjoint action of a semisimple element H in $\mathfrak{h}$. Denote by $\chi = (f|\bullet)$ the linear form on $\mathfrak{g}$ associated with f.

Let $\mathfrak{l}$ be an isotropic subspace of the symplectic space $(\mathfrak{g}_1, \omega)$. Let $N_{\mathfrak{l}}$ be the unipotent subgroup of G whose Lie algebra is $\mathfrak{n}_{\mathfrak{l}} = \mathfrak{l}^{\perp,\omega} \oplus \mathfrak{g}_{\geqslant 2}$. Consider the quantum comoment map $\widehat{\mu}_{\mathfrak{l}} : \mathfrak{n}_{\mathfrak{l}} \to \mathcal{U}(\mathfrak{g}) \otimes_{\mathbf{C}} \mathcal{W}(\mathfrak{l}^{\perp,\omega}/\mathfrak{l})$ defined by the formula (4.2.2.1). Theorem 4.4.2.2 implies the following statement.

Proposition 4.4.3.1. *There is an equivalence of categories*

$$\begin{tikzcd} \mathsf{Wh}(\mathcal{U}(\mathfrak{g}) \otimes_{\mathbf{C}} \mathcal{W}(\mathfrak{l}^{\perp,\omega}/\mathfrak{l}), \mathfrak{n}_{\mathfrak{l}}, \widehat{\mu}_{\mathfrak{l}}) \arrow[r, bend left] \arrow[r, phantom, "\simeq"] & \mathcal{U}(\mathfrak{g}, f)\text{-}\mathsf{Mod}. \arrow[l, bend left] \end{tikzcd}$$

Consider the particular case when $\mathfrak{l}$ is a Lagrangian subspace of $\mathfrak{g}_1$. Then the symplectic space $\mathfrak{l}^{\perp,\omega}/\mathfrak{l}$ is trivial and the coadjoint orbit of $\overline{\chi}_{\mathfrak{l}} := \chi|_{\mathfrak{n}_{\mathfrak{l}}}$ is reduced to one point, that is to say that χ restricts to a character of $\mathfrak{n}_{\mathfrak{l}}$. A $\mathfrak{g}$-module V is called a *Whittaker module* with respect to $\overline{\chi}_{\mathfrak{l}}$ if for any x in $\mathfrak{n}_{\mathfrak{l}}$, the element $x+\chi(x)\,\mathrm{id}_V$ acts locally nilpotently. Denote by $\mathsf{Wh}(\mathfrak{g},\mathfrak{n}_{\mathfrak{l}},\overline{\chi}_{\mathfrak{l}})$ the category of Whittaker modules. The equality

$$\mathsf{Wh}(\mathcal{U}(\mathfrak{g}),\mathfrak{n}_{\mathfrak{l}},\widehat{\mu}_{\mathfrak{l}}) = \mathsf{Wh}(\mathfrak{g},\mathfrak{n}_{\mathfrak{l}},\overline{\chi}_{\mathfrak{l}})$$

holds because $\widehat{\mu}_{\mathfrak{l}}(x) = x+\chi(x)1$ for any x in $\mathfrak{n}_{\mathfrak{l}}$. In this particular case,Proposition 4.4.3.1 states the usual Skryabin equivalence.

Proposition 4.4.3.2 ([Skr02])**.** *There is an equivalence of categories*

$$\begin{tikzcd} \mathsf{Wh}(\mathfrak{g},\mathfrak{n}_{\mathfrak{l}},\overline{\chi}_{\mathfrak{l}}) \arrow[r, bend left] \arrow[r, phantom, "\simeq"] & \mathcal{U}(\mathfrak{g},f)\text{-}\mathsf{Mod}. \arrow[l, bend left] \end{tikzcd}$$

For $i=1,2$, let f_i be a nilpotent element and H_i be in the Cartan subalgebra $\mathfrak{h}$. Assume that the Lie algebra grading $\mathfrak{g} = \bigoplus_{\delta\in\mathbf{Z}} \mathfrak{g}_\delta^{(i)}$ defined by the adjoint action of H_i is a good grading for f_i. Since $[H_1,H_2]=0$, the good gradings define a bigrading $\mathfrak{g} = \bigoplus_{\delta_1,\delta_2\in\mathbf{Z}} \mathfrak{g}_{\delta_1,\delta_2}$. Set $f_0 := f_2 - f_1$. Assume that the conditions ($\bigstar$) holds and consider the quantum comoment map

$$\widehat{\mu}_0 : \mathfrak{n}_0 \longrightarrow \mathcal{U}(\mathfrak{g},f_1), \quad x \longmapsto x\otimes 1 + \overline{\chi_0}1 \bmod \widehat{I}_1$$

given by Theorem 4.3.1.3. There is also a natural algebra isomorphism

$$(\mathcal{U}(\mathfrak{g},f_1)/\widehat{I}_0)^{N_0} \cong \mathcal{U}(\mathfrak{g},f_2).$$

A $\mathcal{U}(\mathfrak{g},f_1)$-module V is Whittaker if for any x in $\mathfrak{n}_0$, $x+\chi_2(x)\,\mathrm{id}_V$ acts locally nilpotently. Theorem 2.3.3.3 allow us to apply Theorem 4.4.2.2 to the quantum comoment map $\widehat{\mu}_0 : \mathfrak{n}_0 \to \mathcal{U}(\mathfrak{g},f_1)$.

Proposition 4.4.3.3 ([GJ24, Main Theorem 4])**.** *There is an equivalence of categories*

$$\begin{tikzcd} \mathsf{Wh}(\mathcal{U}(\mathfrak{g},f_1),\mathfrak{n}_0,\widehat{\mu}_0) \arrow[r, bend left] \arrow[r, phantom, "\simeq"] & \mathcal{U}(\mathfrak{g},f_2)\text{-}\mathsf{Mod}. \arrow[l, bend left] \end{tikzcd}$$

5 - BRST cohomology

5.1. Vertex algebras and the geometry of arc spaces

In this section, we give some definitions and properties about vertex algebras, vertex Poisson algebras and their relation to arc spaces of Poisson varieties. As a preliminary, we recall the notions of vector superspaces and formal distributions. Our main references are [Kac98, FBZ04, DSK06, Ara15, AM25]. We end the section by making some remarks about affine GIT quotients of arc spaces.

5.1.1. Vector superspaces

A *vector superspace* is a $\mathbf{Z}/2\mathbf{Z}$-graded vector space $V = V_0 \oplus V_1$. The elements that are homogenous for the $\mathbf{Z}/2\mathbf{Z}$-grading are called *parity-homogeneous* and the degree of such an element v is called its *parity*, denoted by $|v|$. An element is called *even* if its parity is 0, *odd* if its parity is 1. The tensor product of two superspaces is naturally a superspace.

We recall that a *superalgebra* $A = A_0 \oplus A_1$ is a vector superspace equipped with a $\mathbf{C}$-linear multiplication

$$A \otimes_{\mathbf{C}} A \longrightarrow A, \quad a \otimes b \longmapsto ab$$

that is even. If a and b are parity-homogeneous elements in A, their *superbracket* is defined by $[a, b] \coloneqq ab - (-1)^{|a||b|} ba$.

A linear map $\phi : V \to W$ between two superspaces V, W is called *even* if the inclusions

$$\phi(V_0) \subseteq W_0 \quad \text{and} \quad \phi(V_1) \subseteq W_1$$

hold, *odd* if the inclusions

$$\phi(V_0) \subseteq W_1 \quad \text{and} \quad \phi(V_1) \subseteq W_0$$

hold. This implies that the space of linear maps between two superspaces is itself a superspace.

Denote by $\operatorname{End} V$ the algebra of linear endomorphisms on a vector space V. If V is a superspace, then $\operatorname{End} V$ is a superalgebra for the multiplication given by the composition.

5.1.2. Formal distributions

Let V be a vector space. The space of *formal distributions with coefficients in V* is defined by

$$V[\![z, z^{-1}]\!] \coloneqq \Big\{ \sum_{n \in \mathbf{Z}} v_n z^n \mid v_\bullet \in V^{\mathbf{Z}} \Big\},$$

where z is a formal variable. We also need to consider the analogue with a second formal variable w:

$$V[\![z, z^{-1}, w, w^{-1}]\!] := \Big\{ \sum_{m,n \in \mathbf{Z}} v_{m,n} z^m w^n \mid v_{\bullet,\bullet} \in V^{\mathbf{Z} \times \mathbf{Z}} \Big\}.$$

We define the formal derivative of $\sum_{n \in \mathbf{Z}} v_n z^n$, a formal distribution with coefficients in V, by

$$\partial_z \Big(\sum_{m \in \mathbf{Z}} v_m z^m \Big) := \sum_{m \in \mathbf{Z}} m v_m z^{m-1}.$$

The space of *(formal) Laurent series*, denoted by $V(\!(z)\!)$, is the subspace of formal distributions $\sum_{n \in \mathbf{Z}} v_n z^n$ with coefficients in V such that $v_n = 0$ for n smaller than some negative integer. The space of *formal series*, denoted by $V[\![z]\!]$, is the subspace of formal distributions $\sum_{n \in \mathbf{Z}} v_n z^n$ with coefficients in V such that $v_n = 0$ for all negative indices n.

If A is a (possibly nonassociative) algebra, then $A(\!(z)\!)$ and $A[\![z]\!]$ are algebras for the multiplication defined by

$$\Big(\sum_{n \in \mathbf{Z}} a_n z^n \Big) \Big(\sum_{n \in \mathbf{Z}} b_n z^n \Big) := \sum_{n \in \mathbf{Z}} \Big(\sum_{i+j=n} a_i b_j \Big) z^n,$$

where $\sum_{n \in \mathbf{Z}} a_n z^n, \sum_{n \in \mathbf{Z}} b_n z^n$ are in $A(\!(z)\!)$. Note that the sums $\sum_{i+j=n} a_i b_j$ involves finitely many nonzero terms for all n in $\mathbf{Z}$.

Remark 5.1.2.1. In general, if $\sum_{n \in \mathbf{Z}} a_n z^n, \sum_{n \in \mathbf{Z}} b_n z^n$ belong to $A[\![z, z^{-1}]\!]$, then the sum $\sum_{i+j=n} a_i b_j$ can involve infinitely many nonzero terms and is not well-defined. Hence, $A[\![z, z^{-1}]\!]$ is not an algebra.

If A is a algebra, it is possible to multiply two formal distributions of two different variables, $a(z) = \sum_{m \in \mathbf{Z}} a_m z^m$ and $b(w) = \sum_{n \in \mathbf{Z}} b_n w^n$, to get a formal distribution

$$a(z)b(w) := \sum_{m,n \in \mathbf{Z}} a_m b_n z^m w^n$$

of two variables.

If $V = V_0 \oplus V_1$ is a superspace, then the decomposition

$$V[\![z, z^{-1}]\!] = V_0[\![z, z^{-1}]\!] \oplus V_1[\![z, z^{-1}]\!]$$

makes the space of formal distributions of one variable a superspace. In the same way, the space of formal distributions of two variables, the space of Laurent series and the space of formal series are superspaces.

5.1.3. Graded vertex superalgebras

Definition 5.1.3.1. *A* vertex superalgebra *is a vector superspace* $\mathcal{V} = \mathcal{V}_0 \oplus \mathcal{V}_1$ *equipped with the data of*

1. *a nonzero even element* $\mathbf{1}$ *in* $\mathcal{V}$*, called the* vacuum vector*,*
2. *an even linear operator* $\partial : \mathcal{V} \to \mathcal{V}$*, called the* translation operator*,*
3. *an even linear operator*

$$\mathcal{V} \otimes_{\mathbf{C}} \mathcal{V} \longrightarrow \mathcal{V}(\!(z)\!), \quad a \otimes b \longmapsto a(z)b := \sum_{n \in \mathbf{Z}} a_{(n)}b\, z^{-n-1},$$

called the state-field correspondence*,*

such that the following axioms are satisfied:

1. *for any* a *in* $\mathcal{V}$*, one has*

$$\mathbf{1}(z)a = a \quad \textit{and} \quad a(z)\mathbf{1} = a + zp(z)$$

for some $p(z)$ *in* $\mathcal{V}[\![z]\!]$*,*

2. *for any* a *in* $\mathcal{V}$*,* $[\partial, a(z)] = \partial_z a(z)$ *as elements in* $(\operatorname{End}\mathcal{V})[\![z, z^{-1}]\!]$*,*
3. *for any parity-homogeneous elements* a, b *in* $\mathcal{V}$*, there is some positive integer* k *depending on* a *and* b *such that*

$$(z-w)^k[a(z), b(w)] = 0,$$

where $[\bullet, \bullet]$ *denotes the superbracket in* $(\operatorname{End}\mathcal{V})[\![z, w, z^{-1}, w^{-1}]\!]$*.*

A vertex algebra *is a purely even vertex superalgebra, that is to say that all elements in the underlying superspace are even.*

For a in a vertex superalgebra $\mathcal{V}$, the formal distribution

$$a(z) := \sum_{n \in \mathbf{Z}} a_{(n)}\, z^{-n-1}$$

with coefficients in $\operatorname{End}\mathcal{V}$ is called the *field associated with* a. For b in $\mathcal{V}$, the coefficient $a_{(n)}b$ is called the n-th product of a and b. For all n bigger than some integer depending on a and b, $a_{(n)}b = 0$.

For two vectors a, b in $\mathcal{V}$, their *normally ordered product* is defined by

$$:\!ab\!: \;:=\; a_{(-1)}b.$$

For a, b, c in $\mathcal{V}$, we adopt the convention: $:\!abc\!: \;:=\; :\!a(:\!bc\!:)\!:$. For n a nonnegative integer, one has the relation

$$a_{(-n-1)}b = \frac{1}{n!}:\!\partial^n(a)b\!:.$$

The translation operator ∂ is a derivation for the normally ordered product:

$$\partial({:}ab{:}) = {:}\partial(a)b{:} + {:}a\partial(b){:}.$$

The λ-*superbracket* is the formal polynomial defined by

$$[a_\lambda b] \coloneqq \sum_{n=0}^{\infty} a_{(n)}b\,\frac{\lambda^n}{n!}.$$

The λ-superbracket takes values in $V \otimes_{\mathbf{C}} \mathbf{C}[\lambda]$ and it controls the nonassociativity and the noncommutativity of the normally ordered product.

Remark 5.1.3.2. We can think about a vertex superalgebra $\mathcal{V}$ as a noncommutative and nonassociative superalgebra (for the normally ordered product) equipped with a derivation and a λ-bracket satisfying some compatibility conditions.

Let $\{a_i\}_{i=1}^m$ be a set of parity-homogeneous vectors in $\mathcal{V}$. Denote by $\leqslant$ the lexicographic order on $\{1, \ldots, m\} \times \mathbf{Z}_{\geqslant 0}$. The vertex superalgebra $\mathcal{V}$ is said to be *freely generated* by the *strong generators* $\{a_i\}_{i=1}^m$ if the normally ordered products

$${:}\partial^{n_1}(a_{i_1}) \cdots \partial^{n_k}(a_{i_k}){:} \quad \text{for} \quad \begin{cases} k \geqslant 0 \quad \text{an integer}, \\ i_\bullet \in \{1, \ldots, m\}^k \quad \text{and} \quad n_\bullet \in (\mathbf{Z}_{\geqslant 0})^k, \\ (i_j, k_j) \leqslant (i_{j+1}, k_{j+1}) \quad \text{if} \quad a_{i_j} \quad \text{is even}, \\ (i_j, k_j) < (i_{j+1}, k_{j+1}) \quad \text{if} \quad a_{i_j} \quad \text{is odd}, \end{cases}$$

form a basis of $\mathcal{V}$ as a vector space.

Let $\mathcal{V}, \mathcal{W}$ are two vertex superalgebras. An even linear map $\phi : \mathcal{V} \to \mathcal{W}$ is called a *vertex superalgebra homomorphism* if the following identities hold:

$$\phi(\mathbf{1}) = \mathbf{1}, \quad \partial \circ \phi = \phi \circ \partial, \quad \phi(a_{(n)}b) = \phi(a)_{(n)}\phi(b),$$

for all a, b in $\mathcal{V}$ and all integers n.

Example 5.1.3.3. Recall that the *Virasoro Lie algebra* is the vector space

$$\mathfrak{vir} \coloneqq \mathbf{C}\mathbf{1} \oplus \bigoplus_{n \in \mathbf{Z}} \mathbf{C}L_n$$

equipped with the Lie bracket defined by imposing that $\mathbf{1}$ is a central element and by the relations

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3 - m}{12}\delta_{m+n=0}\mathbf{1}$$

for all m, n in $\mathbf{Z}$.

For a complex number c, denote by $\mathbf{C}|c\rangle$ the one-dimensional module over the Lie subalgebra

$$\mathfrak{vir}_{\geqslant -1} := \mathbf{C}\mathbf{1} \oplus \bigoplus_{n\geqslant -1} \mathbf{C}L_n$$

defined by

$$L_n|c\rangle = 0 \quad \text{for} \quad n \geqslant -1, \quad \text{and} \quad \mathbf{1}|c\rangle = c|c\rangle.$$

Then the induced module

$$\mathcal{V}ir^c := \mathcal{U}(\mathfrak{vir}) \otimes_{\mathcal{U}(\mathfrak{vir}_{\geqslant -1})} \mathbf{C}|c\rangle$$

is equipped with a vertex algebra structure and is called the *Virasoro vertex algebra* of central charge c.

More precisely, the vacuum vector is $\mathbf{1} := 1 \otimes |c\rangle$ and $\mathcal{V}ir^c$ is freely generated by only one element, which is $L := L_{-2}\mathbf{1}$. The associated field is

$$L(z) = \sum_{n\in\mathbf{Z}} L_n\, z^{-n-2}.$$

The λ-bracket is given by

$$[L_\lambda L] = (\partial + 2\lambda)L + \frac{c}{12}\lambda^3.$$

The translation operator coincides with L_{-1} and $\mathcal{V}ir^c$ is $\mathbf{Z}$-graded by the action of the semisimple operator L_0.

Let $\mathcal{V}$ be a vertex superalgebra. An even vector L in $\mathcal{V}$, with associated field $L(z) = \sum_{n\in\mathbf{Z}} L_n\, z^{-n-2}$, is called a *conformal vector* if the operator L_0 is semisimple on $\mathcal{V}$, if the operator L_{-1} coincides with the translation operator ∂ and if there is a complex number c, called the *central charge*, such that

$$[L_\lambda L] = (\partial + 2\lambda)L + \frac{c}{12}\lambda^3.$$

In particular, the existence of a conformal vector induces a vertex superalgebra homomorphism $\mathcal{V}ir^c \to \mathcal{V}$.

If a in $\mathcal{V}$ is an eigenvector for L_0, the corresponding eigenvalue is denoted by $\Delta(a)$. One can check that for any a, b in $\mathcal{V}$ and any n in $\mathbf{Z}$,

$$\Delta(a_{(n)}b) = \Delta(a) + \Delta(b) - n - 1. \tag{5.1.3.4}$$

The decomposition in eigenspaces,

$$\mathcal{V} = \bigoplus_{\Delta\in\mathbf{C}} \mathcal{V}(\Delta), \quad \text{where} \quad \mathcal{V}(\Delta) := \{a \in \mathcal{V} \mid L_0 a = \Delta a\},$$

makes $\mathcal{V}$ a $\mathbf{C}$-graded vector space.

It can happen that a vertex superalgebra has a grading satisfying (5.1.3.4), but without the existence of a conformal vector (for example, take the vertex algebra $\mathscr{V}^{-h^\vee}(\mathfrak{g})$ defined in Remark 5.2.1.1). A semisimple even linear operator $\mathscr{H}$ on $\mathscr{V}$ is called *Hamiltonian* if it satisfies, for any a in $\mathscr{V}$, the relation

$$[\mathscr{H}, a(z)] = z\partial_z a(z) + (\mathscr{H}a)(z).$$

If a is an eigenvector for $\mathscr{H}$, the corresponding eigenvalue is denoted by $\Delta(a)$ and the identity (5.1.3.4) holds. The decomposition in eigenspaces is denoted by

$$\mathscr{V} = \bigoplus_{\Delta\in\mathbf{C}} \mathscr{V}(\Delta), \quad \text{where} \quad \mathscr{V}(\Delta) \coloneqq \{a \in \mathscr{V} \mid \mathscr{H}a = \Delta a\}.$$

Definition 5.1.3.5. *In this thesis, we will call* graded vertex superalgebra *a vertex superalgebra equipped with a Hamiltonian operator whose eigenvalues lie in the discrete set $\frac{1}{K}\mathbf{Z}$ for some integer $K \geqslant 1$.*

5.1.4. Poisson vertex algebra associated with the Li filtration

We recall that a *superalgebra* R is called *supercommutative* if for any pair of parity-homogeneous elements a, b in R, $ab = (-1)^{|a||b|}ba$. A *Poisson vertex superalgebra* is a supercommutative associative unital superalgebra R equipped with a $\mathbf{C}$-linear even derivation $\partial : R \to R$ and an even operator

$$R \otimes_{\mathbf{C}} R \longrightarrow R \otimes_{\mathbf{C}} \mathbf{C}[\lambda], \quad a \otimes b \longmapsto \{a_\lambda b\} = \sum_{n=0}^{\infty} a_{\{n\}} b\, \frac{\lambda^n}{n!},$$

called the *Poisson λ-superbracket*, satisfying the following identifies

$$\begin{aligned}
\{\partial a_\lambda b\} &= -\lambda\{a_\lambda b\},\\
\partial\{a_\lambda b\} &= \{(\partial a)_\lambda b\} + \{a_\lambda \partial b\},\\
\{a_\lambda b\} &= -(-1)^{|a||b|}\{b_{-\lambda-\partial} a\},\\
\{\{a_\lambda b\}_{\lambda+\mu} c\} &= \{a_\lambda\{b_\mu c\}\} - (-1)^{|a||b|}\{b_\mu\{a_\lambda c\}\},\\
\{a_\lambda bc\} &= \{a_\lambda b\}c + (-1)^{|a||b|} b\{a_\lambda c\},
\end{aligned}$$

for parity-homogeneous elements a, b, c in R.

Let R, S are two Poisson vertex superalgebras. A linear map $\phi : R \to S$ is called a *Poisson vertex superalgebra homomorphism* if it is even and if the following identities hold:

$$\phi(\mathbf{1}) = \mathbf{1}, \quad \partial \circ \phi = \phi \circ \partial, \quad \phi(ab) = \phi(a)\phi(b), \quad \phi(a_{\{n\}} b) = \phi(a)_{\{n\}}\phi(b),$$

for all a, b in R and all nonnegative integers n.

Given a vertex superalgebra $\mathscr{V}$, one can define its Li filtration $\mathrm{F}^{\bullet}_{\mathrm{Li}}\mathscr{V}$ in the following way [Li05]. For an integer $p \leqslant 0$, $\mathrm{F}^{p}_{\mathrm{Li}}\mathscr{V}$ is equal to $\mathscr{V}$. For $p \geqslant 0$, the linear subspace $\mathrm{F}^{p}_{\mathrm{Li}}\mathscr{V}$ is spanned by the elements of the form

$$:\!\partial^{n_1}(a_1)\cdots\partial^{n_k}(a_k)\!: \quad \text{where} \quad \left\{ \begin{array}{ll} k \geqslant 0, \quad a_\bullet \in \mathscr{V}^k \quad \text{and} \quad n_\bullet \in (\mathbf{Z}_{\geqslant 0})^k, \\ \text{so that} \quad \sum_{i=1}^k n_i \geqslant p \end{array}\right.$$

It is a nonincreasing filtration of $\mathscr{V}$ satisfying the following properties:

$$\mathrm{F}^0_{\mathrm{Li}}\, V = V, \quad \partial(\mathrm{F}^p_{\mathrm{Li}}\, \mathscr{V}) \subseteq \mathrm{F}^{p+1}_{\mathrm{Li}}\, \mathscr{V},$$

$$(\mathrm{F}^p_{\mathrm{Li}}\, \mathscr{V})_{(n)}(\mathrm{F}^q_{\mathrm{Li}}\, \mathscr{V}) \subseteq \begin{cases} \mathrm{F}^{p+q-n-1}_{\mathrm{Li}}\, \mathscr{V} & \text{for} \quad n \in \mathbf{Z}, \\ \mathrm{F}^{p+q-n}_{\mathrm{Li}}\, \mathscr{V} & \text{for} \quad n \in \mathbf{Z}_{\geqslant 0}. \end{cases} \tag{5.1.4.1}$$

Denote by $[\bullet]_p : \mathrm{F}^p_{\mathrm{Li}}\, \mathscr{V} \twoheadrightarrow \mathrm{F}^p_{\mathrm{Li}}\, \mathscr{V}/\, \mathrm{F}^{p+1}_{\mathrm{Li}}\, \mathscr{V}$ the canonical projection. It is known that the associated graded space

$$\mathrm{gr}^{\mathrm{Li}}_{\bullet}\, \mathscr{V} := \bigoplus_{p \in \mathbf{Z}} \mathrm{F}^p_{\mathrm{Li}}\, \mathscr{V}/\, \mathrm{F}^{p+1}_{\mathrm{Li}}\, \mathscr{V}$$

is a differential supercommutative superalgebra defined by

$$[a]_p \cdot [b]_q := [{:}ab{:}]_{p+q}, \quad \partial[a]_p := [\partial a]_{p+1},$$

where a is in $\mathrm{F}^p_{\mathrm{Li}}\, \mathscr{V}$ and b is in $\mathrm{F}^q_{\mathrm{Li}}\, \mathscr{V}$. By [Li05, Proposition 2.13], $\mathrm{gr}^{\mathrm{Li}}_{\bullet}\, \mathscr{V}$ has the structure of a vertex Poisson superalgebra whose Poisson λ-bracket is given by:

$$\{[a]_{p\lambda}[b]_q\} = \sum_{n=0}^{\infty} [a_{(n)}b]_{p+q-n} \frac{\lambda^n}{n!}.$$

5.1.5. Arc spaces of Poisson varieties

In this section, all superalgebras are assumed to be supercommutative, associative and unital. We refer to [AM25, Chapter 1] or [CLNS18, Chapter 3] for an introduction to jet spaces, arc spaces and their relation to differential algebras. We recall the constructions that we need.

A *differential superalgebra* is a superalgebra S equipped with an even derivation $\partial : S \to S$. Let R be a superalgebra. The *universal differential algebra* spanned by R is a differential superalgebra (R_∞, ∂) equipped with a superalgebra homomorphism $R \to R_\infty$ such that the following universal property holds. For any differential superalgebra (S, ∂') and for any superalgebra homomorphism $\phi : R \to S$, there exists a unique superalgebra homomorphism $\phi_\infty : R_\infty \to S$, intertwining the derivations ($\phi_\infty \circ \partial = \partial' \circ \phi_\infty$), such that the following diagram commutes:

$$\begin{tikzcd} R_\infty \arrow[rr, "\phi_\infty"] & & S \\ & R. \arrow[ul] \arrow[ur, "\phi"'] & \end{tikzcd}$$

Proposition 5.1.5.1. *If R is of finite type, then R_∞ exists and $R \to R_\infty$ is injective.*

For any algebra homomorphism $\phi : R \to S$, there is a unique superalgebra homomorphism $\phi_\infty : R_\infty \to S_\infty$, intertwining the derivations, such that the *following diagram commutes:*

$$\begin{tikzcd}
R_\infty \arrow[r, "\phi_\infty"] & S_\infty \\
R \arrow[u, hook] \arrow[r, "\phi"'] & S. \arrow[u, hook]
\end{tikzcd}$$

Proof. If R is free then R is of the form

$$R = \mathrm{Sym}(V_0) \otimes \bigwedge(V_1),$$

that is to say the tensor product of the symmetric algebra over a finite dimensional vector space V_0 and the exterior algebra over a finite dimensional vector space V_1. Then take

$$R_\infty \coloneqq \mathrm{Sym}(V_0[t^{-1}]t^{-1}) \otimes_{\mathbf{C}} \bigwedge(V_1[t^{-1}]t^{-1}),$$

where $V_i[t^{-1}]t^{-1} \coloneqq V_i \otimes_{\mathbf{C}} \mathbf{C}[t^{-1}]t^{-1}$ for $i = 1, 2$. The operator $-\partial_t$, the opposite derivation with respect to the formal variable t, induces a derivation ∂ on R_∞. There is a natural algebra homomorphism $R \to R_\infty$ determined by

$$V_0 \oplus V_1 \longrightarrow R_\infty, \quad v \longmapsto v \otimes t^{-1}.$$

Consider a superalgebra homomorphism $\phi : R \to S$, where S is equipped with an even derivation ∂'. Then there is one and only one differential superalgebra homomorphism $\phi_\infty : (R_\infty, \partial) \to (S, \partial')$ that makes the diagram commute: it is defined for any F in R and any nonnegative integer n by

$$\phi_\infty(F \otimes t^{-n-1}) = n!(\partial')^n(\phi(F)).$$

In general, if R is finitely generated then R is the quotient

$$R = \big(\mathrm{Sym}(V_0) \otimes_{\mathbf{C}} \bigwedge(V_1)\big)/I$$

of a free superalgebra by some parity-graded ideal I. Denote by I_∞ the ideal in

$$\mathrm{Sym}(V_0[t^{-1}]t^{-1}) \otimes_{\mathbf{C}} \bigwedge(V_1[t^{-1}]t^{-1})$$

spanned by the $\partial^n F$ for F in I and n a nonnegative integer. Then

$$R_\infty \coloneqq \big(\mathrm{Sym}(V_0[t^{-1}]t^{-1}) \otimes_{\mathbf{C}} \bigwedge(V_1[t^{-1}]t^{-1})\big)/I_\infty$$

inherits a derivation and satisfies the universal property as before.

The injectivity of $R \to R_\infty$ follows from the universal property applied to $S = R$ equipped with the zero derivation.

The existence of $\phi_\infty : R_\infty \to S_\infty$, for all algebra homomorphism $\phi : R \to S$, follows from the universal property, as well as the uniqueness. $\square$

Corollary 5.1.5.2. *If R is a superalgebra of finite type, the universal differential superalgebra spanned by R is in fact a colimit of subalgebras of finite type:*

$$R_\infty = \operatorname{colim}_{m \geqslant 0} R_m,$$

where R_m is the subalgebra of R_∞ spanned by the subspaces $\partial^i(R)$ for the integers $0 \leqslant i \leqslant m$.

Proof. As before, denote

$$R = \big(\operatorname{Sym}(V_0) \otimes \bigwedge(V_1)\big)/I.$$

Then is is clear that

$$R_m = \left(\operatorname{Sym}\left(\bigoplus_{i=0}^{m} V_0 t^{-1-i}\right) \otimes_{\mathbf{C}} \bigwedge\left(\bigoplus_{i=0}^{m} V_1 t^{-1-i}\right)\right)/I_m,$$

where I_m is the ideal spanned by the $\partial^n F$ for F in I and $0 \leqslant i \leqslant m$, that is exactly the intersection of I_∞ with R_m.

The truncation of powers of t give injective homomorphisms

$$R_m \hookrightarrow R_{m'} \hookrightarrow R_\infty$$

for all $m \leqslant m'$. So we get a natural map

$$\operatorname{colim}_{m \geqslant 0} R_m \longrightarrow R_\infty.$$

It is injective because all the maps defining the colimit are injective and it is clearly surjective.

□

Poisson vertex algebras arise naturally as the universal differential algebras spanned by Poisson algebras.

Proposition 5.1.5.3 ([Ara12, Proposition 2.3.1])**.** *If R is a finitely generated Poisson superalgebra, then R_∞ has a unique vertex Poisson structure whose Poisson λ-superbracket is given for two elements F_1, F_2 in $R \hookrightarrow R_\infty$ by*

$$\{F_{1\lambda} F_2\} \coloneqq \{F_1, F_2\},$$

where $\{\bullet, \bullet\}$ is the Poisson superbracket on R.

Assume R is purely even of finite type, then it is the coordinate ring of an affine scheme of finite type, $X \coloneqq \operatorname{Spec} R$. We call *arc space* of X the following affine scheme (in general of infinite type):

$$\mathrm{J}_\infty X \coloneqq \operatorname{Spec} R_\infty.$$

For m a nonnegative integer, the *m-jet space* of X is the affine scheme

$$\mathrm{J}_m X \coloneqq \operatorname{Spec} R_m,$$

where R_m is defined in Corollary 5.1.5.2.

The construction of jet spaces and arc spaces is functorial in the category of affine schemes. In particular, if $\phi : X \to Y$ is an affine scheme homomorphism, we denote by

$$\mathrm{J}_m\phi : \mathrm{J}_m X \longrightarrow \mathrm{J}_m Y \quad \text{and} \quad \mathrm{J}_\infty\phi : \mathrm{J}_\infty X \longrightarrow \mathrm{J}_\infty Y$$

the induced maps at the level of m-jets and arcs. This functors also commute with Cartesian products. We recall below their interpretation in term of functors of points.

Proposition 5.1.5.4. *Let X be an affine scheme of finite type. For any $\mathbf{C}$-algebra A, we have some natural bijections*

$$\mathrm{J}_m X(A) \cong X(A[t]/(t^{m+1})) \quad \textit{and} \quad \mathrm{J}_\infty X(A) \cong X(A[\![t]\!]),$$

where $A[t]/(t^{m+1}) \coloneqq A \otimes_{\mathbf{C}} \mathbf{C}[t]/(t^{m+1})$.

Proof. First assume that $X = \operatorname{Spec} R$, where R is a free algebra:

$$R = \operatorname{Sym}(V) \quad \text{and} \quad R_\infty = \operatorname{Sym}(V[t^{-1}]t^{-1}),$$

where V is a finite dimensional vector space. For any $\mathbf{C}$-algebra A, the algebra homomorphisms $R \to A[\![t]\!]$ are determined by their values on the generating subspace V, so there are linear isomorphisms

$$X(A[\![t]\!]) \cong V^* \otimes_{\mathbf{C}} A[\![t]\!] \cong (V^* \otimes_{\mathbf{C}} A)[\![t]\!],$$

where the second isomorphism is due to the finite dimension of V.

Any element in $X(A[\![t]\!])$ acts on $V \otimes_{\mathbf{C}} A[t^{-1}]t^{-1}$ by the following bilinear pairing:

$$(\phi \otimes a(t)) \cdot (v \otimes b(t)) \coloneqq \phi(v) \operatorname{res}_{t=0}(a(t)b(t)), \tag{5.1.5.5}$$

where ϕ is in V^*, v is in V, $a(t)$ is in $A[\![t]\!]$, $b(t)$ is in $A[t^{-1}]t^{-1}$ and

$$\operatorname{res}_{t=0}\Big(\sum_{n\in\mathbf{Z}} c_n t^n\Big) \coloneqq c_{-1}, \quad \text{for} \quad \sum_{n\in\mathbf{Z}} c_n t^n \in A(\!(t)\!).$$

So any element in $X(A[\![t]\!])$ induces an algebra homomorphism $R_\infty \to A$.

Consider an algebra homomorphism $\Phi : R_\infty \to A$. It is determined by its values on

$$V[t^{-1}]t^{-1} = \bigoplus_{n=0}^{\infty} V t^{-n-1},$$

so it corresponds to the action of an element $p(t) = \sum_{n=0}^{\infty} p_n t^n$ in

$$X(A[\![t]\!]) \cong (V^* \otimes_{\mathbf{C}} A)[\![t]\!]$$

whose coefficients are the linear maps

$$p_n : V \longrightarrow A, \quad v \longmapsto \Phi(v \otimes t^{-n-1}).$$

This construction provides the explicit bijection

$$\mathrm{J}_\infty X(A) \cong X(A[\![t]\!]).$$

In the general case, when

$$R = \mathrm{Sym}(V)/I \quad \text{and} \quad R_\infty = \mathrm{Sym}(V[t^{-1}]t^{-1}),$$

where I is some ideal in $\mathrm{Sym}\, V$ and I_∞ is the ideal spanned by the $\partial^n F$ for F in I and $\partial = -\partial_t$. Note that an element F in I is seen as an element in $\mathrm{Sym}(V[t^{-1}]t^{-1})$ because of the inclusion $\mathrm{Sym}\, V \hookrightarrow \mathrm{Sym}(V[t^{-1}]t^{-1})$. In particular, X is a closed subscheme of V^* given by the equations in I.

The $A[\![t]\!]$-points of X are

$$X(A[\![t]\!]) = \{x(t) \in (V^* \otimes_{\mathbf{C}} A)[\![t]\!] \mid \text{for all } F \in I,\ F(x(t)) = 0\}.$$

Let us make sense of the equation $F(x(t)) = 0$ in this context. Consider the polynomial $F = v_1 \dots v_k$, where $v_1, \dots, v_k$ are in V. Its evaluation at $x(t)$, where

$$x(t) = \sum_{n=0}^{\infty} \phi_n \otimes a_n\, t^n, \quad \text{for} \quad \phi_n \in V^* \quad \text{and} \quad a_n \in A,$$

is given by (we omit the $\otimes$ symbols):

$$\begin{aligned} v_1 \dots v_k(x(t)) &= \sum_{n=0}^{\infty} \Big(\sum_{i_1 + \cdots i_k = n} \phi_{i_1}(v_1) \cdots \phi_{i_k}(v_k) a_{i_1} \cdots a_{i_k} \Big) t^n \\ &= \sum_{n=0}^{\infty} \mathrm{res}_{t=0} \left(\sum_{i_1 + \cdots i_k = n} (t^{-1-i_1} v_1 \cdots t^{-1-i_k} v_k) \Big(\sum_{i=0}^{\infty} \phi_i a_i t^i \Big) \right) t^n \\ &= \sum_{n=0}^{\infty} \mathrm{res}_{t=0} \Big((-\partial_t)^n (t^{-1} v_1 \cdots t^{-1} v_k) \Big(\sum_{i=0}^{\infty} \phi_i a_i t^i \Big) \Big) t^n. \end{aligned}$$

So for a general F in $\mathrm{Sym}\, V$, we have

$$F(x(t)) = \sum_{n=0}^{\infty} \mathrm{res}_{t=0} \big(\partial^n F(x(t)) \big) t^n,$$

where F is identified with its image in $\mathrm{Sym}(V[t^{-1}]t^{-1})$.

Recall that an element of $x(t)$ in $(V^* \otimes_{\mathbf{C}} A)[\![t]\!]$ corresponds to an element x_∞ in $\mathrm{J}_\infty V^*(A)$, that is to say an algebra homomorphism $\mathrm{Sym}(V[t^{-1}]t^{-1}) \to A$, under the residue action defined in (5.1.5.5). Hence, the equality

$$\mathrm{res}_{t=0}\left(\partial^n F(x(t))\right) = \partial^n F(x_\infty),$$

holds in A, where the right-hand side is the evaluation of the polynomial $\partial^n F$ at the point x_∞ in $\mathrm{J}_\infty V^*(A)$.

So, under the identification $(V^* \otimes_{\mathbf{C}} A)[\![t]\!] \cong \mathrm{J}_\infty V^*(A)$, the equations

$$F(x(t)) = 0 \quad \text{in} \quad A[\![t]\!], \quad \text{for} \quad F \in I,$$

are equivalent to the equations

$$\partial^n F(x_\infty) = 0 \quad \text{in} \quad A, \quad \text{for} \quad F \in I.$$

Because

$$\mathrm{J}_\infty X(A) = \{x_\infty \in \mathrm{J}_\infty V^*(A) \mid \text{for all } F \in I \text{ and } n \geqslant 0,\ \partial^n F(x_\infty) = 0\},$$

we deduce the bijection

$$X(A[\![t]\!]) \cong \mathrm{J}_\infty X(A).$$

For the m jets, the construction is similar and we skip the details. □

Remark 5.1.5.6. For any $\mathbf{C}$-algebra A, there is a natural surjective algebra homomorphism $A[\![t]\!] \twoheadrightarrow A[t]/(t^{m+1})$. It induces a map $\mathrm{J}_\infty X \to \mathrm{J}_m X$, that is the geometric map induced by the inclusion $\mathbf{C}[\mathrm{J}_m X] \hookrightarrow \mathbf{C}[\mathrm{J}_\infty X]$ stated in Corollary 5.1.5.2.

5.1.6. Affine GIT quotient of an arc space

Consider a connected affine algebraic group N and denote by $\mathfrak{n}$ its Lie algebra. For any nonnegative integer m, the jet scheme $\mathrm{J}_m N$ is a connected variety because the group N is smooth and connected [CLNS18, Corollary 3.7.7 and Proposition 4.1.1]. Moreover, because N is an algebraic group, its functor of points takes value in the category of groups, so the functors of points of $\mathrm{J}_m N$ and $\mathrm{J}_\infty N$ are group-valued too. Therefore, $\mathrm{J}_m N$ is an algebraic group and $\mathrm{J}_\infty N$ is a group scheme [Mil17, Section 1.4].

The structure of the Lie algebra of $\mathrm{J}_m N$ is well-known, we give a construction that uses the functor of points of N. For an alternative proof using the Hopf algebra structure of $\mathbf{C}[N]$, see [AM25, Example 1.2].

Lemma 5.1.6.1. *The Lie algebra of* $\mathrm{J}_m N$ *is*

$$\mathfrak{n}[t]/(t^{m+1}) \coloneqq \mathfrak{n} \otimes_{\mathbf{C}} \mathbf{C}[t]/(t^{m+1}),$$

equipped with the $\mathbf{C}[t]$*-linear Lie bracket extending the one on* $\mathfrak{n}$.

Proof. We recall the following useful characterisation of the Lie algebra $\mathfrak{m}$ of an affine algebraic group M, see [Mil17, Section 10.b] for details. For any $\mathbf{C}$-algebra A, the natural projection $A[\varepsilon]/(\varepsilon^2) \to A$ induces a group homomorphism

$$M(A[\varepsilon]/(\varepsilon^2)) \longrightarrow M(A).$$

The kernel

$$\mathfrak{m}(A) := \operatorname{Ker}\Big(M(A[\varepsilon]/(\varepsilon^2)) \to M(A)\Big).$$

is an Abelian subgroup of $M(A[\varepsilon]/(\varepsilon^2))$, its in fact an A-module for the A-action induced by the rescaling ε. The A-module $\mathfrak{m}(A)$ also inherits an A-linear Lie bracket induced by the conjugation on $M(A[\varepsilon]/(\varepsilon^2))$.

By change of basis,the tensor product $A \otimes_{\mathbf{C}} \mathfrak{m}$ is equipped with a natural A-linear Lie bracket. According to [Mil17], the natural inclusion $\mathfrak{m} \hookrightarrow \mathfrak{m}(A)$ induces an A-linear Lie algebra isomorphism

$$A \otimes_{\mathbf{C}} \mathfrak{m} \cong \mathfrak{m}(A).$$

Take $M := \mathrm{J}_m N$ the jet group of an algebraic group N. Because of the above characterisation of Lie algebras of algebraic groups and Proposition 5.1.5.4, the Lie algebra of $\mathrm{J}_m N$ is given as the kernel of

$$N(\mathbf{C}[t,\varepsilon]/(t^{m+1},\varepsilon^2)) \longrightarrow N(\mathbf{C}[t]/(t^{m+1})),$$

that is $\mathfrak{n} \otimes_{\mathbf{C}} \mathbf{C}[t]/(t^{m+1})$. □

Let X be an affine scheme of finite type with an action of a connected affine algebraic group N. For any nonnegative integer m, there is an induced action of $\mathrm{J}_m N$ on $\mathrm{J}_m X$. The infinite type group scheme $\mathrm{J}_\infty N$ acts on the arc space $\mathrm{J}_\infty X$. Denote by $\rho : N \times X \to X$ the action, by $\mathrm{J}_m\rho$ and $\mathrm{J}_\infty\rho$ the induced actions at the level of m-jets and arcs. The subalgebra of $\mathrm{J}_\infty N$-invariant functions is defined by

$$\mathbf{C}[\mathrm{J}_\infty X]^{\mathrm{J}_\infty N} := \{F \in \mathbf{C}[\mathrm{J}_\infty X] \mid \mathrm{J}_\infty\rho^*(F) = 1 \otimes F\},$$

where $\mathrm{J}_\infty\rho^* : \mathbf{C}[\mathrm{J}_\infty X] \to \mathbf{C}[\mathrm{J}_\infty N] \otimes_{\mathbf{C}} \mathbf{C}[\mathrm{J}_\infty X]$ is the corresponding co-action.

By imposing that the ideal $\mathfrak{n}[t]t^{m+1}$ acts by zero, we get an action by derivation of the Lie algebra $\mathfrak{n}[t] := \mathfrak{n} \otimes_{\mathbf{C}} \mathbf{C}[t]$ on each coordinate ring $\mathbf{C}[\mathrm{J}_m X]$ that lifts the $\mathfrak{n}[t]/(t^{m+1})$-action. There is also an induced action of $\mathfrak{n}[t]$ on the colimit

$$\mathbf{C}[\mathrm{J}_\infty X] = \operatorname{colim}_{m \geqslant 0} \mathbf{C}[\mathrm{J}_m X].$$

The following proposition has been stated many times in the literature, see for example [Ara15, BFM23, AM25]. We provide a proof.

Proposition 5.1.6.2. *Let N be a connected affine algebraic group acting on an affine scheme X. Then*

$$\mathbf{C}[\mathrm{J}_\infty X]^{\mathrm{J}_\infty N} = \mathbf{C}[\mathrm{J}_\infty X]^{\mathfrak{n}[t]}.$$

Proof. For any F in $\mathbf{C}[\mathrm{J}_\infty X]$, there is some integer m such that F belongs to the subalgebra $\mathbf{C}[\mathrm{J}_m X]$. Then

$$\mathrm{J}_\infty \rho^*(F) = \mathrm{J}_m \rho^*(F). \tag{5.1.6.3}$$

The relation (5.1.6.3) implies that they are natural embeddings

$$\mathbf{C}[\mathrm{J}_m X]^{\mathrm{J}_m N} \hookrightarrow \mathbf{C}[\mathrm{J}_{m'} X]^{\mathrm{J}_{m'} N} \hookrightarrow \mathbf{C}[\mathrm{J}_\infty X]^{\mathrm{J}_\infty N}$$

for all $m \leqslant m'$, so by universal property there is an embedding

$$\mathrm{colim}_{m \geqslant 0}\, \mathbf{C}[\mathrm{J}_m X]^{\mathrm{J}_m N} \hookrightarrow \mathbf{C}[\mathrm{J}_\infty X]^{\mathrm{J}_\infty N},$$

but this natural map is surjective because of (5.1.6.3). So we have an algebra isomorphism

$$\mathrm{colim}_{m \geqslant 0}\, \mathbf{C}[\mathrm{J}_m X]^{\mathrm{J}_m N} = \mathbf{C}[\mathrm{J}_\infty X]^{\mathrm{J}_\infty N}.$$

Similarly, the subalgebra of $\mathfrak{n}[t]$-invariant functions of $\mathrm{J}_\infty X$ is given by

$$\mathrm{colim}_{m \geqslant 0}\, \mathbf{C}[\mathrm{J}_m X]^{\mathfrak{n}[t]} = \mathbf{C}[\mathrm{J}_\infty X]^{\mathfrak{n}[t]}.$$

By definition of the $\mathfrak{n}[t]$-action, the $\mathfrak{n}[t]/(t^{m+1})$ and $\mathfrak{n}[t]$-invariants are the same:

$$\mathbf{C}[\mathrm{J}_m X]^{\mathfrak{n}[t]/(t^{m+1})} = \mathbf{C}[\mathrm{J}_m X]^{\mathfrak{n}[t]}.$$

Because all the J_m are connected, one has

$$\mathbf{C}[\mathrm{J}_m X]^{\mathrm{J}_m X} = \mathbf{C}[\mathrm{J}_m X]^{\mathfrak{n}[t]}.$$

Taking the colimits, we deduce the desired equality:

$$\mathbf{C}[\mathrm{J}_\infty X]^{\mathrm{J}_\infty N} = \mathbf{C}[\mathrm{J}_\infty X]^{\mathfrak{n}[t]}.$$

□

Remark 5.1.6.4. There is a natural map

$$\mathrm{J}_\infty X /\!/ \mathrm{J}_\infty N \longrightarrow \mathrm{J}_\infty (X/\!/N),$$

but is neither injective or surjective in general [LSS15].

5.2. Examples of vertex superalgebras

We review the vertex algebras that we need to construct affine W-algebras, following [DSK06].

5.2.1. Affine vertex algebras

Let $\mathfrak{a}$ be a finite dimensional Lie algebra and $\kappa : \mathfrak{a} \otimes \mathfrak{a} \to \mathbf{C}$ be an $\mathfrak{a}$-invariant symmetric bilinear form. The affine *Kac–Moody algebra* associated with $\mathfrak{a}$ and κ is defined as the extension of $\mathfrak{a}[t, t^{-1}]$ (with the natural $\mathbf{C}[t, t^{-1}]$-linear bracket) by a central element $\mathbf{1}$, denoted by

$$\widehat{\mathfrak{a}}^{\kappa} \coloneqq \mathfrak{a}[t, t^{-1}] \oplus \mathbf{C}\mathbf{1},$$

with the brackets given by

$$[xt^m, yt^n] = [x, y]t^{m+n} + m\delta_{m+n=0}\kappa(x, y)\,\mathbf{1}$$

for any x, y in $\mathfrak{a}$ and m, n two integers. It contains $\mathfrak{a}[t] \oplus \mathbf{C}\mathbf{1}$ as a Lie subalgebra.

Denote by $\mathbf{C}|1\rangle$ the one-dimensional $\mathfrak{a}[t] \oplus \mathbf{C}\mathbf{1}$-module defined by

$$\mathfrak{a}[t]|1\rangle = 0 \quad \text{and} \quad \mathbf{1}|1\rangle = |1\rangle.$$

The *universal affine vertex algebra* $\mathcal{V}^{\kappa}(\mathfrak{a})$ associated with $\mathfrak{a}$ at level κ is defined as the induced $\widehat{\mathfrak{a}}^{\kappa}$-module

$$\mathcal{V}^{\kappa}(\mathfrak{a}) \coloneqq \mathcal{U}(\widehat{\mathfrak{a}}^{\kappa}) \otimes_{\mathcal{U}(\mathfrak{a}[t] \oplus \mathbf{C}\mathbf{1})} \mathbf{C}|1\rangle.$$

We identify any element x in $\mathfrak{a}$ with its image $xt^{-1} \otimes |1\rangle$ in $\mathcal{V}^{\kappa}(\mathfrak{a})$.

The module $\mathcal{V}^{\kappa}(\mathfrak{a})$ is equipped with a vertex algebra structure such that the field associated with x in $\mathfrak{a}$ is given by the formal distribution

$$x(z) = \sum_{n \in \mathbf{Z}} xt^n\, z^{-n-1},$$

where xt^n is seen as an endomorphism of $\mathcal{V}^{\kappa}(\mathfrak{a})$ by the mean of the $\widehat{\mathfrak{a}}^{\kappa}$-module structure. The vacuum vector is $\mathbf{1} \coloneqq 1 \otimes |1\rangle$. One has the λ-brackets

$$[x_{\lambda} y] = [x, y] + \kappa(x, y)\mathbf{1}\,\lambda \quad \text{for} \quad x, y \in \mathfrak{a}.$$

The vertex algebra $\mathcal{V}^{\kappa}(\mathfrak{a})$ is freely generated by any basis of $\mathfrak{a}$.

Denote by D the derivation on $\mathcal{U}(\widehat{\mathfrak{a}}^{\kappa})$ that acts as the derivation $-t\partial_t$ on $\mathfrak{a}[t, t^{-1}]$ and by 0 on $\mathbf{1}$. The operator $\mathcal{H}^{\mathfrak{a}} \coloneqq D \otimes \mathrm{id}_{\mathbf{C}|1\rangle}$ is a Hamiltonian operator such that the conformal degree of x in $\mathfrak{a}$ is $\Delta(x) = 1$.

Recall that the Lie brackets extends to a Poisson bracket on the symmetric algebra of $\mathfrak{a}$, which is the coordinate ring of the dual space $\mathfrak{a}^*$. There is a Poisson vertex isomorphism $\mathrm{gr}^{\mathrm{Li}}\, \mathcal{V}^{\kappa}(\mathfrak{a}) \cong \mathbf{C}[\mathrm{J}_{\infty}\mathfrak{a}^*]$. The Poisson λ-bracket is given for x, y in $\mathfrak{a}$ by $\{x_{\lambda} y\} = [x, y]$.

Let $\mathfrak{g}$ be a simple finite dimensional complex Lie algebra and $\mathfrak{h}$ be a Cartan subalgebra. Denote by $(\bullet|\bullet)$ the non-degenerate symmetric invariant bilinear form on $\mathfrak{g}$ given by $(\bullet|\bullet) \coloneqq (2h^{\vee})^{-1}\kappa_{\mathfrak{g}}$, where $h^{\vee}$ be the dual Coxeter number of $\mathfrak{g}$ and $\kappa_{\mathfrak{g}}$ is the Killing form of the Lie algebra $\mathfrak{g}$. Then we denote

$$\widehat{\mathfrak{g}}^{k} \coloneqq \widehat{\mathfrak{g}}^{k(\bullet|\bullet)} \quad \text{and} \quad \mathcal{V}^{k}(\mathfrak{g}) \coloneqq \mathcal{V}^{k(\bullet|\bullet)}(\mathfrak{g}).$$

According to [DSK06, Remark 1.23], for any H in the Cartan subalgebra $\mathfrak{h}$, the element $\mathscr{H}^{\mathfrak{g}} + (\partial H)_{(1)}$ defines a Hamiltonian operator vector of $\mathcal{V}^k(\mathfrak{g})$. The conformal degree of x in $\mathfrak{g}$ is $\Delta(x) = 1 - \delta$ if there is a complex number δ such that $[H, x] = \delta x$.

Remark 5.2.1.1 ([DSK06, (1.60)]). In fact, by the Sugawara construction, the grading on $\mathcal{V}^k(\mathfrak{g})$ comes from a conformal vector if and only if $k + h^\vee \neq 0$. The level $k = -h^\vee$, the vertex algebra $\mathcal{V}^{-h^\vee}(\mathfrak{g})$ is not conformal.

For $\mathfrak{n}$ a finite dimensional nilpotent Lie algebra and 0 the trivial bilinear form on $\mathfrak{n}$, we use the notation

$$\mathcal{V}(\mathfrak{n}) \coloneqq \mathcal{V}^0(\mathfrak{n}).$$

5.2.2. Clifford vertex superalgebras

Let V be a finite dimensional vector space equipped with a symmetric bilinear form $\langle \bullet, \bullet \rangle$. The *affine Clifford superalgebra* associated with $(V, \langle \bullet, \bullet \rangle)$ is the associative unital superalgebra, denoted by $\widehat{\mathcal{C}\ell}(V)$, that is defined as the quotient of the tensor algebra $\bigoplus_{k=0}^{\infty} (V[t, t^{-1}])^{\otimes k}$ by the two-sided ideal spanned by

$$xt^m \otimes yt^n + yt^n \otimes xt^m - \delta_{m+n=-1} \langle v, w \rangle 1, \qquad \text{for} \quad x, y \in V \quad \text{and} \quad m, n \in \mathbf{Z}.$$

The $\mathbf{Z}/2\mathbf{Z}$-grading on $\widehat{\mathcal{C}\ell}(V)$ is induced by the reduction modulo 2 of the $\mathbf{Z}$-grading on the tensor algebra. The superalgebra $\widehat{\mathcal{C}\ell}(V)$ contains $V[t, t^{-1}]$ as a generating subspace.

Denote by $\widehat{\mathcal{C}\ell}(V)_{\geqslant 0}$ the subalgebra spanned by $V[t]$, it is commutative. Denote by $\mathbf{C}|1\rangle$ the trivial one-dimensional module over $\widehat{\mathcal{C}\ell}(V)_{\geqslant 0}$. The *Clifford vertex superalgebra* associated with $(V, \langle \bullet, \bullet \rangle)$, also referred to as the vertex superalgebra of charged fermions in [KRW03, KW04], is defined as the induced $\widehat{\mathcal{C}\ell}(V)$-module

$$\mathcal{F}(V) \coloneqq \widehat{\mathcal{C}\ell}(V) \otimes_{\widehat{\mathcal{C}\ell}(V)_{\geqslant 0}} \mathbf{C}|1\rangle.$$

We identify x in V with $xt^{-1} \otimes |1\rangle$, so x is an odd element in $\mathcal{F}(V)$.

The module $\mathcal{F}(V)$ is equipped with a vertex superalgebra structure such that the field associated with an element x in V is given by

$$x(z) = \sum_{n \in \mathbf{Z}} xt^n \, z^{-n-1}.$$

The vacuum vector is $\mathbf{1} \coloneqq 1 \otimes |1\rangle$. The λ-brackets are given by

$$[x_\lambda y] = \langle x, y \rangle \mathbf{1} \quad \text{for} \quad x, y \in V.$$

The vertex algebra $\mathcal{F}(V)$ is freely generated by any basis of V.

Suppose that $\mathfrak{n}$ is a finite-dimensional vector space with a basis $\{x_i\}_{i=1}^n$ and $\mathfrak{n}^*$ be the dual space with the dual basis $\{\xi_i\}_{i=1}^n$. The vector space $\mathfrak{n} \oplus \mathfrak{n}^*$ is equipped with a symmetric bilinear form induced by the parity pairing, and we can consider the corresponding Clifford vertex superalgebra $\mathcal{F}(\mathfrak{n} \oplus \mathfrak{n}^*)$. We use the notations

$$\phi_x := xt^{-1} \otimes |1\rangle \quad \text{and} \quad \phi^*_\xi := \xi t^{-1} \otimes |1\rangle$$

for x in $\mathfrak{n}$ and ξ in $\mathfrak{n}^*$. The vertex superalgebra $\mathcal{F}(\mathfrak{n} \oplus \mathfrak{n}^*)$ is freely generated by the odd elements

$$\phi_i := \phi_{x_i} \quad \text{and} \quad \phi^*_i := \phi_{\xi_i},$$

where $1 \leqslant i \leqslant d$. The λ-brackets are given for x, y in $\mathfrak{n}$ and ξ, η in $\mathfrak{n}^*$ by

$$[\phi_x{}_\lambda \phi^*_\xi] = \xi(x), \quad [\phi_x{}_\lambda \phi_y] = [\phi^*_\xi{}_\lambda \phi^*_\eta] = 0.$$

The Clifford vertex superalgebra $\mathcal{F}(\mathfrak{n} \oplus \mathfrak{n}^*)$ is graded by the so-called *charge*:

$$\mathcal{F}^\bullet(\mathfrak{n} \oplus \mathfrak{n}^*) = \bigoplus_{n \in \mathbf{Z}} \mathcal{F}^n(\mathfrak{n} \oplus \mathfrak{n}^*),$$

$$\text{where} \quad \begin{cases} \operatorname{charge}(\mathbf{1}) := 0, \\ \operatorname{charge}(\partial^i(\phi_x)) := -1 \quad \text{for} \quad x \in \mathfrak{n}, \\ \operatorname{charge}(\partial^i(\phi^*_\xi)) := 1 \quad \text{for} \quad \xi \in \mathfrak{n}^* \quad \text{and} \quad i \geqslant 0. \end{cases}$$

The charge is additive for the normally ordered products:

$$\operatorname{charge}({:}\phi\phi'{:}) = \operatorname{charge}(\phi) + \operatorname{charge}(\phi') \quad \text{for} \quad \phi, \phi' \in \mathcal{F}^\bullet(\mathfrak{n} \oplus \mathfrak{n}^*).$$

The exterior algebra $\bigwedge(\mathfrak{n} \oplus \mathfrak{n}^*)$ is a Poisson superalgebra whose superbracket is given for x, y in $\mathfrak{n}$ and ξ, η in $\mathfrak{n}^*$ by

$$\{\phi_x, \phi^*_\xi\} = \xi(x) \quad \text{and} \quad \{\phi_x, \phi_y\} = \{\phi^*_\xi, \phi^*_\eta\} = 0.$$

The universal differential superalgebra $\bigwedge_\infty(\mathfrak{n} \oplus \mathfrak{n}^*)$ spanned by $\bigwedge(\mathfrak{n} \oplus \mathfrak{n}^*)$ is a Poisson vertex superalgebra and it has an analogue charge grading. There is a Poisson vertex superalgebra isomorphism

$$\operatorname{gr}^{\mathrm{Li}} \mathcal{F}^\bullet(\mathfrak{n} \oplus \mathfrak{n}^*) \cong \textstyle\bigwedge^\bullet_\infty(\mathfrak{n} \oplus \mathfrak{n}^*).$$

This isomorphism respects the charge grading.

For each $1 \leqslant i \leqslant n$, fix a complex number m_i and set $H^{\mathcal{F}}_i(z) := -{:}\phi^*_i(z)\phi_i{:}$ in $\mathcal{F}(\mathfrak{n} \oplus \mathfrak{n}^*)$. These elements satisfy the relations $[H^{\mathcal{F}}_i{}_\lambda H^{\mathcal{F}}_j] = \delta_{i=j}\lambda$. Then, according to [DSK06, Remark 1.23], the element

$$L^{\mathcal{F}}(m_\bullet) := L^{\mathcal{F}} + \sum_{i=1}^n m_i \partial H^{\mathcal{F}}_i$$

is a conformal vector of $\mathcal{F}(\mathfrak{n} \oplus \mathfrak{n}^*)$. The corresponding central charge is equal to $c^{\mathcal{F}}(m_\bullet) = \sum_{i=1}^n \left(-12m_i^2 + 12m_i - 2\right)$. The conformal degrees are given by

$$\Delta(\phi_i) = 1 - m_i \quad \text{and} \quad \Delta(\phi^*_i) = m_i \quad \text{for} \quad 1 \leqslant i \leqslant d.$$

5.2.3. Weyl vertex algebras

Let (V, ω) be a finite-dimensional symplectic space. The affine Weyl algebra associated with (V, ω) is the associative unital algebra, denoted by $\widehat{\mathscr{W}}(V)$, that is defined as the quotient of the tensor algebra $\bigoplus_{k=0}^{\infty}(V[t,t^{-1}])^{\otimes k}$ by the two-sided ideal spanned by

$$vt^m \otimes wt^n - wt^n \otimes vt^m - \delta_{m+n=-1}\omega(v,w)1, \qquad \text{for} \quad x, y \in V \quad \text{and} \quad m, n \in \mathbf{Z}.$$

It contains $V[t, t^{-1}]$ as a generating subspace.

Denote by $\widehat{\mathscr{W}}(V)_{\geqslant 0}$ the subalgebra spanned by $V[t]$, it is commutative. Denote by $\mathbf{C}|1\rangle$ the trivial one-dimensional module over $\widehat{\mathscr{W}}(V)_{\geqslant 0}$. The *Weyl vertex algebra* associated with (V, ω), also referred to as the vertex superalgebra of neutral fermions in [KRW03, KW04] or $\beta\gamma$-system, is defined as the induced $\widehat{\mathscr{W}}(V)$-module

$$\mathscr{A}(V) \coloneqq \widehat{\mathscr{W}}(V) \otimes_{\widehat{\mathscr{W}}(V)_{\geqslant 0}} \mathbf{C}|1\rangle.$$

Denote $\psi_v \coloneqq vt^{-1} \otimes |1\rangle$ for any v in V.

The module $\mathscr{A}(V)$ is equipped with a vertex algebra such that the field associated with ψ_v, for v in V, is given by

$$\psi_v(z) = \sum_{n \in Z} vt^n \, z^{-n-1}.$$

The vacuum vector is $\mathbf{1} \coloneqq 1 \otimes |1\rangle$. The λ-brackets are given by

$$[\psi_{v\,\lambda} \psi_w] = \omega(v,w)\mathbf{1} \quad \text{for} \quad v, w \in V.$$

If $\{v_i\}_{i=1}^{2s}$ is a basis of V, then $\mathscr{A}(V)$ is freely generated by the

$$\psi_i \coloneqq \psi_{v_i}, \quad \text{for} \quad 1 \leqslant i \leqslant 2s.$$

There is a vertex Poisson algebra isomorphism

$$\operatorname{gr}^{\mathrm{Li}} \mathscr{A}(V) \cong \mathbf{C}[\mathrm{J}_\infty V],$$

where the Poisson bracket on the right-hand side is induced by the symplectic form ω, see Example 2.1.2.3.

Let $\{v^i\}_{i=1}^{2s}$ be the dual basis of V with respect to ω, so that $\omega(v_i, v^j) = \delta_{i=j}$, and set $\psi^i \coloneqq \psi_{v^i}$. Then the element $L^{\mathscr{A}} \coloneqq \frac{1}{2}\sum_{i=1}^{2s} {:}\partial(\psi^i)\psi_i{:}$ is a conformal vector of $\mathscr{A}(V)$ and the conformal degree of ψ_v is $\Delta(\psi_v) = \frac{1}{2}$ for any v in V.

More generally, assume that there is a symplectic isomorphism of V with the cotangent bundle $\mathrm{T}^* W$ of a Lagrangian subspace W of V. Assume that the vector $\{v_i\}_{i=1}^{s}$ form a basis of W, so $\{v^i\}_{i=1}^{s}$ is a basis of W^*. Set $H_i^{\mathscr{A}} = -{:}\psi^i\psi_i{:}$

for $1 \leqslant i \leqslant s$. Their λ-brackets are $[H_i^{\mathscr{A}}{}_\lambda H_j^{\mathscr{A}}] = -\delta_{i=j}\lambda$. For each $1 \leqslant i \leqslant s$, fix a complex number a_i. Then, the element

$$L^{\mathscr{A}}(a_\bullet) \coloneqq L^{\mathscr{A}} + \sum_{i=1}^{s} a_i \partial(H_i^{\mathscr{A}})$$

defines a conformal vector of $\mathscr{A}(V)$, and $c^{\mathscr{A}}(a_\bullet) = -\frac{1}{2}\dim V + 12\sum_{i=1}^{s} {a_i}^2$ is the corresponding central charge [DSK06, Remark 1.23]. Then the conformal degree are given by

$$\Delta(\psi_i) = \frac{1}{2} - a_i \quad \text{and} \quad \Delta(\psi^i) = \frac{1}{2} + a_i$$

for $1 \leqslant i \leqslant s$.

5.3. Vanishing theorems for BRST cohomology

In this section, we define the BRST cohomology in the context of algebraic group action on a Poisson variety with a moment map. The historical reference is [KS87]. There is an analogue for the arc space of a Poisson variety, and we state a vanishing result for this BRST cohomology (Theorem 5.3.2.1). We also define BRST cochain complex in the vertex algebra context. Good references are [Ara15, AKM15, AM25]. In Theorem 6 (5.3.5.1), we give a sufficient condition for the BRST cohomology to vanish except in degree 0 and to compute the graded Poisson vertex algebra constructed from the Li filtration on this cohomology.

5.3.1. BRST cohomology for Poisson varieties

Let N be a connected affine algebraic group acting on a Poisson variety X by Poisson automorphism, M be a connected closed normal subgroup of N and $\mu : X \to \mathfrak{m}^*$ be an N-equivariant moment map.

Consider a Poisson variety $\widetilde{X}$ endowed with an action of the group N and a moment map $\widetilde{\mu} : \widetilde{X} \to \mathfrak{n}^*$. In addition, assume the existence of a dominant N-equivariant Poisson map $\rho : \widetilde{X} \to X$ such that the following square commutes:

(5.3.1.1)
$$\begin{tikzcd} \widetilde{X} \arrow[r, "\tilde{\mu}"] \arrow[d, "\rho"'] & \mathfrak{n}^* \arrow[d, two heads] \\ X \arrow[r, "\mu"'] & \mathfrak{m}^*. \end{tikzcd}$$

The following construction is taken from [KS87]. Denote by $\{x_i\}_{i \in I(\mathfrak{n})}$ a basis of the vector space $\mathfrak{n}$. Consider the graded Poisson superalgebra

$$\widetilde{C}^\bullet \coloneqq \mathbf{C}[\widetilde{X}] \otimes_{\mathbf{C}} \bigwedge\nolimits^\bullet(\mathfrak{n} \oplus \mathfrak{n}^*),$$

with the notations introduced in Section 5.2.2. It contains the following element:

$$\widetilde{Q} \coloneqq \sum_{i \in I(\mathfrak{n})} \widetilde{\mu}^*(x_i)\phi^i - \frac{1}{2}\sum_{i,j \in I(\mathfrak{n})} \phi_{[x_i,x_j]}\phi_i^*\phi_j^*, \tag{5.3.1.2}$$

which is of charge one, and so odd. Consider the Poisson adjoint action of $\widetilde{Q}$, denoted by $\widetilde{d} \coloneqq \{\widetilde{Q}, \bullet\}$. The pair $(\widetilde{C}^\bullet, \widetilde{d})$ forms a cochain complex and the associated cohomology $\mathrm{H}^\bullet(\widetilde{C}^\bullet, \widetilde{d})$ has a natural structure of graded Poisson superalgebra.

By definition, the dominant map $\rho : \widetilde{X} \to X$ induces a Poisson algebra embedding $\mathbf{C}[X] \hookrightarrow \mathbf{C}[\widetilde{X}]$. For simplicity, we identify the left-hand side algebra to its image in the right-hand side. Under this identification, we can define the following graded Poisson super-subalgebra of $\widetilde{C}$,

$$C^\bullet \coloneqq \mathbf{C}[X] \otimes_{\mathbf{C}} \textstyle\bigwedge^\bullet(\mathfrak{m} \oplus \mathfrak{n}^*),$$

where $\bigwedge^\bullet(\mathfrak{m} \oplus \mathfrak{n}^*)$ denotes the Poisson super-subalgebra spanned by the elements ϕ_x and ϕ_ξ^*, where x is in $\mathfrak{m}$ and ξ is in $\mathfrak{n}^*$. The super-subalgebra C is stable by the differential $\widetilde{d}$.

Denote by d the restriction of the differential $\widetilde{d}$ to the superalgebra C. Then the pair $(C^\bullet, d)$ forms a cochain complex, called the *BRST cochain complex* associated with the Poisson variety X, the acting group N and the moment map $\mu : X \to \mathfrak{m}^*$. Its homology $\mathrm{H}^\bullet(\widetilde{C}^\bullet, \widetilde{d})$ has a natural structure of graded Poisson superalgebra. We refer to [AM25, Theorem 7.1] for the following statement.

Theorem 5.3.1.3. *Let X be a Poisson variety with an action of the group N by Poisson automorphism and an N-equivariant moment map $\mu : X \to \mathfrak{m}^*$ for the action of the normal subgroup M of N, and define the corresponding BRST complex (C, d). Make the following assumptions:*

1. *the moment map $\mu : X \to \mathfrak{m}^*$ is smooth,*
2. *there exists a closed subvariety S of $\mu^{-1}(0)$ such that the action map*

$$\alpha : N \times S \longrightarrow \mu^{-1}(0), \quad (g, x) \longmapsto g \cdot x$$

is an isomorphism.

Then the subvariety S is isomorphic to $\mu^{-1}(0)//N$, so the variety S inherits a Poisson structure. Moreover, there is a natural Poisson isomorphism

$$\mathrm{H}^\bullet(C, d) \cong \mathrm{H}^\bullet(\mathfrak{n}, \mathbf{C}[N]) \otimes_{\mathbf{C}} \mathbf{C}[S],$$

where the right-hand side is the tensor product of the trivial Poisson superalgebra $\mathrm{H}^\bullet(\mathfrak{n}, \mathbf{C}[N])$ given by the Lie algebra cohomology of the $\mathfrak{n}$-module $\mathbf{C}[N]$ (the action is by left-invariant derivations) and the Poisson algebra $\mathbf{C}[S]$.

The Lie algebra $\mathfrak{n}$ is finite dimensional and the Lie algebra cohomology of the module $\mathbf{C}[N]$ is defined as the cohomology of the associated Chevalley–Eilenberg complex, see [Wei97, Section 7.7] for details.

Remark 5.3.1.4. In [AM25], a proof is provided for $M = N$ by using a spectral sequence argument : the BRST cohomology is the total cohomology of a double complex mixing Koszul homology and Lie algebra cohomology. This argument also works in the case $M \neq N$.

5.3.2. BRST cohomology for arc spaces

Let $\rho : \widetilde{X} \to X$ be a Poisson N-equivariant dominant map between two affine Poisson varieties $\widetilde{X}, X$ equipped with algebraic N-actions by Poisson automorphism. Let $\widetilde{\mu} : \widetilde{X} \to \mathfrak{n}^*$ and $\mu : X \to \mathfrak{m}^*$ be two N-equivariant moment maps such that (5.3.1.1) commutes.

The Poisson superalgebra $\widetilde{C}^\bullet = \mathbf{C}\big[\widetilde{X}\big] \otimes_{\mathbf{C}} \bigwedge^\bullet(\mathfrak{n} \oplus \mathfrak{n}^*)$ induces a Poisson vertex superalgebra

$$\widetilde{C}^\bullet_\infty \coloneqq \mathbf{C}\big[\mathrm{J}_\infty \widetilde{X}\big] \otimes_{\mathbf{C}} \textstyle\bigwedge^\bullet_\infty(\mathfrak{n} \oplus \mathfrak{n}^*),$$

in the notations introduced in Section 5.2.2. It contains the element $\widetilde{Q}$, introduced in (5.3.1.2), which is of charge degree one, and so odd. Consider the adjoint action of $\widetilde{Q}$, denoted by $\widetilde{d}_\infty \coloneqq \big\{\widetilde{Q}_\lambda \bullet \big\}_{\lambda=0}$. The pair $\big(\widetilde{C}^\bullet_\infty, \widetilde{d}_\infty\big)$ forms a cochain complex and its associated cohomology $\mathrm{H}^\bullet\big(\widetilde{C}^\bullet_\infty, \widetilde{d}_\infty\big)$ has a natural structure of graded Poisson vertex superalgebra.

By definition, the dominant map $\rho : \widetilde{X} \to X$ induces a Poisson vertex algebra embedding $\mathbf{C}[\mathrm{J}_\infty X] \hookrightarrow \mathbf{C}\big[\mathrm{J}_\infty \widetilde{X}\big]$, so to simplify, we identify the left-hand side algebra to its image in the right-hand side. Under this identification, we can define the following graded Poisson super-subalgebra of $\widetilde{C}_\infty$,

$$C^\bullet_\infty \coloneqq \mathbf{C}[\mathrm{J}_\infty X] \otimes_{\mathbf{C}} \textstyle\bigwedge^\bullet_\infty(\mathfrak{m} \oplus \mathfrak{n}^*),$$

where $\bigwedge^\bullet_\infty(\mathfrak{m} \oplus \mathfrak{n}^*)$ denotes the Poisson vertex super-subalgebra spanned by elements $\partial^n \phi_x$ and $\partial^n \phi^*_\xi$ for x in $\mathfrak{m}$, ξ in $\mathfrak{n}^*$ and n a nonnegative integer. The super-subalgebra C_∞ is stable by the differential $\widetilde{d}_\infty$.

Denote by d_∞ the restriction of the differential $\widetilde{d}_\infty$ to the superalgebra C_∞. Then the pair (C_∞, d_∞) forms a cochain complex, called the *Poisson vertex BRST cochain complex* associated with the arc space of the Poisson variety X with its action of N and the moment map $\mu : X \to \mathfrak{m}^*$. Its cohomology $\mathrm{H}^\bullet(C_\infty, d_\infty)$ has a natural structure of graded Poisson superalgebra. We refer to [AM25, Theorem 9.2] for the following statement.

Theorem 5.3.2.1. *Let X be a Poisson variety with an action of the group N by Poisson automorphism and an N-equivariant moment map $\mu : X \to \mathfrak{m}^*$ for the action of the normal subgroup M of N, and define the corresponding BRST complex (C_∞, d_∞). Make the following assumptions:*

1. *the moment map $\mu : X \to \mathfrak{m}^*$ is smooth,*

2. *there exists a closed subvariety S of $\mu^{-1}(0)$ such that the action map*

$$\alpha : N \times S \longrightarrow \mu^{-1}(0), \quad (g, x) \longmapsto g \cdot x$$

is an isomorphism.

Then there is a natural Poisson vertex isomorphism

$$\mathrm{H}^\bullet(C_\infty, d_\infty) \cong \mathrm{H}^\bullet(\mathfrak{n}[t], \mathbf{C}[\mathrm{J}_\infty N]) \otimes_{\mathbf{C}} \mathbf{C}[\mathrm{J}_\infty S],$$

where the right-hand side is the tensor product of the trivial Poisson vertex superalgebra $\mathrm{H}^\bullet(\mathfrak{n}[t], \mathbf{C}[\mathrm{J}_\infty N])$ given by the cohomology of the $\mathfrak{n}[t]$-module $\mathbf{C}[\mathrm{J}_\infty N]$ and the Poisson vertex algebra $\mathbf{C}[\mathrm{J}_\infty S]$.

Because $\mathfrak{n}[t]$ is infinite-dimensional, one must explain the meaning of the cohomology $\mathrm{H}^\bullet(\mathfrak{n}[t], \mathbf{C}[\mathrm{J}_\infty N])$. For any nonnegative integer m, the Lie algebra $\mathfrak{n}[t]/(t^{m+1})$ is finite dimensional and the Lie algebra cohomology of the module $\mathbf{C}[\mathrm{J}_m N]$ is defined as the cohomology of the associated Chevalley–Eilenberg complex,

$$C^\bullet_{\mathrm{CE},m} \coloneqq \mathbf{C}[\mathrm{J}_m X] \otimes_{\mathbf{C}} \bigwedge\nolimits^\bullet \left((\mathfrak{n}[t]/(t^{m+1}))^*\right) \cong \mathbf{C}[\mathrm{J}_m X] \otimes_{\mathbf{C}} \bigwedge\nolimits^\bullet_m(\mathfrak{n}^*),$$

where $\bigwedge_m(\mathfrak{n}^*)$ denotes the m-jets of $\bigwedge(\mathfrak{n}^*)$.

The colimit of these complexes is a cochain complex

$$C^\bullet_{\mathrm{CE},\infty} \coloneqq \mathrm{colim}_{m \geqslant 0}\, C^\bullet_{\mathrm{CE},m} \cong \mathbf{C}[\mathrm{J}_\infty X] \otimes_{\mathbf{C}} \bigwedge\nolimits^\bullet_\infty(\mathfrak{n}^*),$$

that is a subcomplex of the usual Chevalley–Eilenberg associated to the $\mathfrak{n}[t]$-module $\mathbf{C}[\mathrm{J}_\infty N]$. By definition,

$$\mathrm{H}^\bullet(\mathfrak{n}[t], \mathbf{C}[\mathrm{J}_\infty N]) \coloneqq \mathrm{H}^\bullet(C^\bullet_{\mathrm{CE},\infty}).$$

Remark 5.3.2.2. Theorem 5.3.2.1 is slightly different than the statement given in [AM25]. First, we deal with the case when M and N are not necessarily equal, see Remark 5.3.1.4.

Moreover, in [AM25], the hypotheses of the theorem are less restrictive: the moment maps $\mu : X \to \mathfrak{m}^*$ and $\mathrm{J}_\infty\mu : \mathrm{J}_\infty X \to \mathrm{J}_\infty\mathfrak{m}^*$ are assumed to be flat. We claim that if μ is smooth, then $\mathrm{J}_\infty\mu$ is automatically flat. Indeed, if μ is smooth, then $\mathrm{J}_m\mu : \mathrm{J}_m X \to \mathrm{J}_m\mathfrak{m}^*$ is flat for any integer $m \geqslant 0$ [CLNS18, Proposition 3.7.4]. By using the equational criterion for flatness [Eis13, Corollary 6.5], we deduce that $\mathrm{J}_\infty\mu$ is flat too.

5.3.3. BRST cohomology for vertex algebras

Let M, N be two connected affine algebraic groups such that M is a normal subgroup of N, and denote by $\mathfrak{m}, \mathfrak{n}$ their respective Lie algebras. We assume from now that the groups M and N are *unipotent*, in particular the Killing forms of their Lie algebras are zero. Let $\mathcal{V}, \widetilde{\mathcal{V}}$ be two vertex algebras such that $\mathcal{V}$ is a subalgebra of $\widetilde{\mathcal{V}}$.

Let $\mathcal{V}(\mathfrak{m}), \mathcal{V}(\mathfrak{n})$ be the universal affine vertex algebras associated respectively with $\mathfrak{m}, \mathfrak{n}$. Let $\widetilde{\Upsilon} : \mathcal{V}(\mathfrak{n}) \to \widetilde{\mathcal{V}}$ be a vertex algebra homomorphism, it induces an $\mathfrak{n}[t]$-action on $\widetilde{\mathcal{V}}$. We assume that the vertex subalgebra $\mathcal{V}$ is a $\mathfrak{n}[t]$-submodule of $\widetilde{\mathcal{V}}$. Moreover, we assume the inclusion $\widetilde{\Upsilon}(\mathcal{V}(\mathfrak{m})) \subseteq \mathcal{V}$. Whence, by restriction of $\widetilde{\Upsilon}$, we get a vertex algebra map $\Upsilon : \mathcal{V}(\mathfrak{m}) \to \mathcal{V}$ such that the following square is commutative:

(5.3.3.1)
$$\begin{tikzcd} \mathcal{V}(\mathfrak{n}) \arrow[r, "\widetilde{\Upsilon}"] & \widetilde{\mathcal{V}} \\ \mathcal{V}(\mathfrak{m}) \arrow[u, hook] \arrow[r, "\Upsilon"'] & \mathcal{V}. \arrow[u, hook] \end{tikzcd}$$

In particular, the vertex algebra $\mathcal{V}$ is a module over the Lie algebra $\mathfrak{n}[t] \oplus \mathfrak{m}[t^{-1}]$. The homomorphism $\Upsilon : \mathcal{V}(\mathfrak{m}) \to \mathcal{V}$ is called a *chiral comoment map*.

Let $\{x_i\}_{i \in I(\mathfrak{n})}$ be a basis of $\mathfrak{n}$. Let $\mathcal{F}^\bullet(\mathfrak{m} \oplus \mathfrak{n}^*)$ be the graded vertex subalgebra of the Clifford vertex algebra $\mathcal{F}^\bullet(\mathfrak{n} \oplus \mathfrak{n}^*)$ strongly generated by elements ϕ_x and ϕ^*_ξ, where x is in $\mathfrak{m}$ and ξ is in $\mathfrak{n}^*$. Consider the charge-graded vertex superalgebra
$$\widetilde{\mathcal{C}}^\bullet \coloneqq \widetilde{\mathcal{V}} \otimes_{\mathbf{C}} \mathcal{F}^\bullet(\mathfrak{n} \oplus \mathfrak{n}^*).$$
It contains the element
$$\widetilde{Q} \coloneqq \sum_{i \in I(\mathfrak{n})} {:}\widetilde{\Upsilon}(x_i)\phi^*_i{:} - \frac{1}{2} \sum_{i,j \in I(\mathfrak{n})} {:}\phi_{[x_i,x_j]}\phi^*_i\phi^*_j{:}$$
which is of charge one and odd. Consider the operator given by 0-th mode of $\widetilde{Q}$, denoted by $\widetilde{d} \coloneqq \widetilde{Q}_{(0)} = \big[\widetilde{Q}\,{}_\lambda\, \bullet\,\big]_{\lambda=0}$. The pair $\big(\widetilde{\mathcal{C}}^\bullet, \widetilde{d}\big)$ forms a cochain complex and its associated cohomology $\mathrm{H}^\bullet\big(\widetilde{\mathcal{C}}^\bullet, \widetilde{d}\big)$ has a natural structure of graded vertex superalgebra.

Define the following graded vertex super-subalgebra
$$\mathcal{C}^\bullet \coloneqq \mathcal{V} \otimes_{\mathbf{C}} \mathcal{F}^\bullet(\mathfrak{m} \oplus \mathfrak{n}^*)$$
of the cochain complex $\widetilde{\mathcal{C}}$.

Lemma 5.3.3.2. *The super-subalgebra $\mathcal{C}$ is stable by the differential $\widetilde{d}$.*

Proof. Using the inclusions $[\mathfrak{n}, \mathfrak{m}] \subseteq \mathfrak{m}$ and $\widetilde{\Upsilon}(\mathcal{V}(\mathfrak{m})) \subseteq \mathcal{V}$, and the fact that $\mathcal{V}$ is a $\mathfrak{n}[t]$-submodule of $\widetilde{\mathcal{V}}$, the lemma follows from a computation on generators. □

Denote by d the restriction of the differential $\widetilde{d}$ to the vertex superalgebra C. Then the pair $(C^\bullet, d)$ forms a cochain complex, called the *BRST cochain complex* associated with the vertex algebra V, equipped with an action of $\mathfrak{n}[t]$, and the chiral comoment map $\Upsilon : V(\mathfrak{m}) \to V$.

5.3.4. Nonnegatively graded quotient complex

Assume that the vertex algebra $\widetilde{V}$ is graded in the sense of Definition 5.1.3.5 and that V is a graded subalgebra. For any element $x \in \widetilde{V}$, denote by x_Δ the image of x through the natural projection $\widetilde{V} \twoheadrightarrow \widetilde{V}(\Delta)$.

Lemma 5.3.4.1. *Suppose that the grading of $\widetilde{V}$ and V is over $\frac{1}{K}\mathbf{Z}_{\geqslant 0}$. Assume that, for any element x in $\mathfrak{n}$, the image $\widetilde{\Upsilon}(x)$ lies in $\bigoplus_{\Delta\leqslant 1} \widetilde{V}(\Delta)$.*

Then we have the following λ-bracket for all x, y in $\mathfrak{n}$:

$$[\widetilde{\Upsilon}(x)_{1\lambda}\widetilde{\Upsilon}(y)_1] = \widetilde{\Upsilon}([x,y])_1.$$

Proof. Set $X := \widetilde{\Upsilon}(x)$ and $Y := \widetilde{\Upsilon}(y)$ in $\widetilde{V}$. The elements X and Y belong to $\bigoplus_{\Delta\leqslant 1} \widetilde{V}(\Delta)$ so they decompose as $X = \sum_{i=0}^{\infty} X_{1-\frac{i}{K}}$ and $Y = \sum_{j=0}^{\infty} X_{1-\frac{j}{K}}$. By (5.1.3.4), one has

$$\Delta\big(X_{1-\frac{i}{K}(n)}Y_{1-\frac{j}{K}}\big) = 1 - \frac{i+j}{K} - n$$

that equals 1 if and only if $i = j = n = 0$.

Set $Z := \widetilde{\Upsilon}([x,y])$. Because $[x_\lambda y] = [x,y]$, one has

$$Z = \widetilde{\Upsilon}([x_\lambda y]) = [X_\lambda Y] = X_{(0)}Y,$$

the last equality follows since Z does not contain terms of strictly positive degrees in the formal variable λ.

Comparing the degrees, we get the desired equality: $Z_1 = X_{1(0)}Y_1$. □

When the hypotheses of Lemma 5.3.4.1 holds, there is a vertex algebra homomorphism $\widetilde{\Upsilon}_{\mathrm{st}} : V(\mathfrak{n}) \to \widetilde{V}$ such that $\widetilde{\Upsilon}_{\mathrm{st}}(x) := \widetilde{\Upsilon}(x)_1$ for x in $\mathfrak{n}$. This homomorphism is called the *standard comoment map* associated with Υ. Set

$$\widetilde{Q}_{\mathrm{st}} := \sum_{i\in I(\mathfrak{n})} {:}\widetilde{\Upsilon}_{\mathrm{st}}(x_i)\phi_i^*{:} - \frac{1}{2}\sum_{i,j\in I(\mathfrak{n})} {:}\phi_{[x_i,x_j]}\phi_i^*\phi_j^*{:} \quad \text{and} \quad \widetilde{d}_{\mathrm{st}} := \widetilde{Q}_{\mathrm{st}(0)}.$$

Denote by d_{st} the restriction of the differential $\widetilde{d}_{\mathrm{st}}$ to the vertex superalgebra C. Then, the pair $(C^\bullet, d_{\mathrm{st}})$ forms a cochain complex, called the *standard BRST cochain complex* by analogy with [FF90, KRW03].

Consider the linear subspace

$$\mathcal{I} := \mathrm{Span}_{\mathbf{C}}\{\phi_{x(-n-1)}c,\ (d\phi_x)_{(-n-1)}c \mid n \geqslant 0,\ x \in \mathfrak{m},\ c \in C\}$$

of the BRST complex $\mathcal{C}$. This subspace is closed by d and so it descends to the quotient $\mathcal{C}_+^\bullet \coloneqq \mathcal{C}^\bullet/\mathcal{I}^\bullet$. Denote by d_+ the induced differential. Hence the pair $(\mathcal{C}_+^\bullet, d_+)$ forms a cochain complex of vector spaces, *with no natural vertex algebra structure* in general.

The following theorem is a generalisation of [AM25, Proposition 9.3].

Theorem 5.3.4.2. *Let N be a unipotent affine algebraic group and M be a unipotent normal subgroup. Let $\widetilde{\mathcal{V}}$ be a nonnegatively graded vertex algebra and $\mathcal{V}$ be a graded vertex subalgebra. Let $\widetilde{\Upsilon} : \mathcal{V}(\mathfrak{n}) \to \widetilde{\mathcal{V}}$ be a vertex algebra homomorphism which restricts to a homomorphism $\Upsilon : \mathcal{V}(\mathfrak{m}) \to \mathcal{V}$. Assume that $\mathcal{V}$ is a $\mathfrak{n}[t]$-submodule of $\widetilde{\mathcal{V}}$. Define the associated BRST cochain complex $(\mathcal{C}^\bullet, d)$ and the quotient complex $(\mathcal{C}_+^\bullet, d_+)$ as above. Assume the following conditions:*

1. *for all Δ in $\frac{1}{K}\mathbf{Z}_{\geqslant 0}$, the homogenous subspace $\widetilde{\mathcal{V}}(\Delta)$ is finite-dimensional,*
2. *for any element x in $\mathfrak{n}$, the image $\widetilde{\Upsilon}(x)$ lies in $\bigoplus_{\Delta \leqslant 1} \widetilde{\mathcal{V}}(\Delta)$,*
3. *the standard chiral comoment map $\widetilde{\Upsilon}_{\mathrm{st}} : \mathcal{V}(\mathfrak{n}) \to \widetilde{\mathcal{V}}$ defined by Lemma 5.3.4.1 induces a free action of the envelopping algebra $\mathcal{U}(\mathfrak{m}[t^{-1}]t^{-1})$ on $\mathcal{V}$.*

Then there is an isomorphism $\mathrm{H}^\bullet(\mathcal{C}^\bullet, d) \cong \mathrm{H}^\bullet(\mathcal{C}_+^\bullet, d_+)$ of vector spaces induced by the canonical projection $\mathcal{C} \twoheadrightarrow \mathcal{C}_+$.

To prove this theorem, we extend the Hamiltonian operator $\mathcal{H}$ of $\widetilde{\mathcal{V}}$ to the BRST complex $\widetilde{\mathcal{C}} = \widetilde{\mathcal{V}} \otimes_{\mathbf{C}} \mathcal{F}(\mathfrak{n} \oplus \mathfrak{n}^*)$ by adding the conformal vector $L_0^{\mathcal{F}}$. The degrees are $\mathcal{H}(\phi_x) = 1$ and $\mathcal{H}(\phi_\xi^*) = 0$ for x in $\mathfrak{n}$ and ξ in $\mathfrak{n}^*$. This grading descends to the subalgebra $\mathcal{C}$ and we denote the homogeneous subspaces by

$$\mathcal{C}(\Delta) \coloneqq \{c \in \mathcal{C} \mid \mathcal{H}(c) = \Delta c\} \quad \text{for} \quad \Delta \in \frac{1}{K}\mathbf{Z}_{\geqslant 0}.$$

One gets an induced decreasing filtration on $\mathcal{C}$ defined by

$$\mathrm{F}^p_{\mathcal{H}}\,\mathcal{C} \coloneqq \bigoplus_{\Delta \leqslant -p/K} \mathcal{C}(\Delta) \quad \text{for} \quad p \in \mathbf{Z}.$$

The filtration is exhaustive and bounded from below, that is to say,

$$\mathcal{C} = \bigcup_{p \in \mathbf{Z}} \mathrm{F}^p_{\mathcal{H}}\,\mathcal{C} \quad \text{and} \quad \mathrm{F}^p_{\mathcal{H}}\,\mathcal{C}_V = 0 \quad \text{for} \quad p > 0.$$

This filtration is preserved by the coboundary operator d by the second assumption of Theorem 5.3.4.2. Therefore the filtered cochain complex $(\mathrm{F}^\bullet_{\mathcal{H}}\,\mathcal{C}^\bullet, d)$ induces a spectral sequence $\{(\mathcal{E}_r, d_r)\}_{r=0}^\infty$ [Wei97, Section 5.4]. The zero page coincides with the standard BRST complex introduced after Lemma 5.3.4.1:

$$\mathcal{E}_0^{p,q} = \mathcal{C}^{p+q}\Big(\frac{1}{K}p\Big), \quad d_0 = d_{\mathrm{st}},$$

for p, q in $\mathbf{Z}$. In particular, the standard differential respects the conformal grading. Whence, the first page is:

$$\mathcal{E}_1^{p,q} = \mathrm{H}^{p+q}\Big(\mathcal{C}^\bullet\Big(\frac{1}{K}p\Big), d_{\mathrm{st}}\Big).$$

The standard comoment map $\Upsilon_{\mathrm{st}} : \mathcal{V}(\mathfrak{m}) \to \mathcal{V}$ induces an action of the universal envelopping algebra $\mathcal{U}(\mathfrak{m}[t^{-1}]t^{-1})$ of the Lie algebra $\mathfrak{m}[t^{-1}]t^{-1}$, denoted by $\cdot_{\mathrm{st}}$.

Lemma 5.3.4.3. *The 1-st page $\mathcal{E}_1 = \mathrm{H}^\bullet(\mathcal{C}^\bullet, d_{\mathrm{st}})$ is isomorphic to Lie algebra cohomology* $\mathrm{H}^\bullet\Big(\mathfrak{n}[t], \mathcal{V}/\big(\mathfrak{m}[t^{-1}]t^{-1} \cdot_{\mathrm{st}} \mathcal{V}\big)\Big)$.

Proof. Apply the co-analog of the Hochschild–Serre spectral sequence [Vor93, Theorem 2.3] on $(\mathcal{C}, d_{\mathrm{st}})$. It corresponds to the filtration defined by

$$\mathrm{HS}^p\mathcal{C}^n := \mathrm{Span}_{\mathbf{C}}\left\{\phi^*_{\xi_1(-m_1-1)}\cdots\phi^*_{\xi_p(-m_p-1)}c \;\middle|\; \begin{array}{l} c \in \mathcal{C}^{n-p} \\ \xi_\bullet \in (\mathfrak{n}^*)^p,\ m_\bullet \in (\mathbf{Z}_{\geqslant 0})^p \end{array}\right\},$$

where n, p are in $\mathbf{Z}$. This filtration is nonincreasing, bounded from above (it means that $\mathrm{HS}^0\mathcal{C}^n = \mathcal{C}^n$), and it is preserved by the standard differential. There is an induced spectral sequence denoted by $\{(\mathcal{E}_{\mathrm{HS},r}, d_{\mathrm{HS},r})\}_{r=0}^\infty$.

The standard BRST complex splits into a direct sum of finite-dimensional subcomplexes because the Hamiltonian grading is preserved by the standard differential and the dimension of each summand is finite as a consequence of the first assumption of Theorem 5.3.4.2. For any Δ in $\frac{1}{K}\mathbf{Z}_{\geqslant 0}$, the induced Hochschild–Serre filtration on $\mathcal{C}(\Delta)$ is finite and the associated spectral sequence $\{(\mathcal{E}_{\mathrm{HS},r}(\Delta), d_{\mathrm{HS},r})\}_{r=0}^\infty$ is convergent [Wei97, Theorem 5.5.1].

The zero page is given by the following vector space isomorphism

$$\mathcal{E}_{\mathrm{HS},0}^{p,q} = \mathrm{gr}_p^{\mathrm{HS}}\,\mathcal{C}^{p+q} \cong \mathcal{V} \otimes_{\mathbf{C}} \textstyle\bigwedge_\infty^{-q}(\mathfrak{m}) \otimes_{\mathbf{C}} \bigwedge_\infty^{p}(\mathfrak{n}^*),$$

for p, q in $\mathbf{Z}$. The first page is given by

$$\mathcal{E}_{\mathrm{HS},1}^{p,q} = \mathrm{H}_{-q}(\mathfrak{m}[t^{-1}]t^{-1}, \mathcal{V}) \otimes_{\mathbf{C}} \textstyle\bigwedge_\infty^{p}(\mathfrak{n}^*),$$

where $\mathrm{H}_\bullet(\mathfrak{m}[t^{-1}]t^{-1}, \mathcal{V})$ is the Lie algebra homology with coefficients in the vector space $\mathcal{V}$ equipped with the $\mathfrak{m}[t^{-1}]t^{-1}$-module structure induced by Υ_{st}.

The action of $\mathcal{U}(\mathfrak{m}[t^{-1}]t^{-1})$ on $\mathcal{V}$ is free by assumption. Hence, this homology given by

$$\mathrm{H}_\bullet(\mathfrak{m}[t^{-1}]t^{-1}, \mathcal{V}) = \delta_{\bullet=0}\,\mathrm{H}_0(\mathfrak{m}[t^{-1}]t^{-1}, \mathcal{V}) = \mathcal{V}/\big(\mathfrak{m}[t^{-1}]t^{-1} \cdot_{\mathrm{st}} \mathcal{V}\big).$$

Therefore, the spectral sequence collapses at the 2-nd page:

$$\mathcal{E}_{\mathrm{HS},2}^{p,q} = \delta_{q=0}\,\mathcal{E}_{\mathrm{HS},2}^{p,0} = \delta_{q=0}\,\mathrm{H}^p\Big(\mathfrak{n}[t], \mathcal{V}/\big(\mathfrak{m}[t^{-1}]t^{-1} \cdot_{\mathrm{st}} \mathcal{V}\big)\Big).$$

The spectral sequence is convergent on each homogeneous component (for the Hamiltonian grading), so the infinity term of the spectral sequence is

$$\mathscr{E}^{p,q}_{\mathrm{HS},\infty} = \delta_{q=0}\ \mathrm{gr}^{\mathrm{HS}}_p\ \mathrm{H}^p(\mathcal{C}^\bullet, d_{\mathrm{st}}).$$

The convergence also implies that the filtration on $\mathrm{H}^\bullet(\mathcal{C}^\bullet, d_{\mathrm{st}})$ is complete on each homogeneous component and then the collapsing implies the equality

$$\mathrm{H}^p(\mathcal{C}^\bullet, d_{\mathrm{st}}) = \mathrm{gr}^{\mathrm{HS}}_p\ \mathrm{H}^p(\mathcal{C}^\bullet, d_{\mathrm{st}}),$$

and it also implies the isomorphism $\mathscr{E}^{p,q}_{\mathrm{HS},\infty} \cong \mathscr{E}^{p,q}_{\mathrm{HS},2}$. Finally, we get the desired isomorphism for all $p \geqslant 0$:

$$\mathrm{H}^p(\mathcal{C}^\bullet, d_{\mathrm{st}}) \cong \mathrm{H}^p\Big(\mathfrak{n}[t], \mathcal{V}/\big(\mathfrak{m}[t^{-1}]t^{-1} \cdot_{\mathrm{st}} \mathcal{V}\big)\Big).$$

□

Proof of Theorem 5.3.4.2. Consider the filtration induced on the quotient complex:

$$\mathrm{F}^p_{\mathscr{H}}\, \mathcal{C} := \mathrm{F}^p_{\mathscr{H}}\, \mathcal{C}_{\mathcal{V}}/(\mathrm{F}^p_{\mathscr{H}}\, \mathcal{C} \cap \mathcal{I}) \quad \text{for} \quad p \in \mathbf{Z}.$$

Then the corresponding spectral sequence $\{(\mathscr{E}_{+,r}, d_{+,r})\}_{r=0}^{\infty}$ converges to $\mathrm{H}^\bullet(\mathcal{C}_+)$.

The graded ideal associated with $\mathcal{I}$ is its standard analogue:

$$\mathrm{gr}^{\mathrm{F}_{\mathscr{H}}}\, \mathcal{I} = \mathcal{I}_{\mathrm{st}} := \mathrm{Span}_{\mathbf{C}}\{\phi_{x(-n-1)}c,\ d_{\mathrm{st}}(\phi_x)_{(-n-1)}c \mid n \geqslant 0,\ x \in \mathfrak{m},\ c \in \mathcal{C}\},$$

which is graded by the Hamiltonian grading. The 0-th page of the spectral sequence coincides with the corresponding quotient of the standard BRST complex:

$$\mathscr{E}^{p,q}_{+,0} = \mathcal{C}^{p+q}\Big(\frac{1}{K}p\Big)/\mathcal{I}_{\mathrm{st}}\Big(\frac{1}{K}p\Big), \quad d_{+,0} = d_{\mathrm{st},+}, \quad \text{for} \quad p, q \in \mathbf{Z}.$$

In particular, it coincides with the following Lie algebra cohomology complex:

$$\mathscr{E}^{p}_{+,0} = \mathcal{V}/\big(\mathfrak{m}[t^{-1}]t^{-1} \cdot_{\mathrm{st}} \mathcal{V}\big) \otimes_{\mathbf{C}} \textstyle\bigwedge^p_\infty(\mathfrak{n}^*).$$

So the first page is the following Lie algebra cohomology:

$$\mathscr{E}^{p}_{+,1} = \mathrm{H}^p\Big(\mathfrak{n}[t], \mathcal{V}/\big(\mathfrak{m}[t^{-1}]t^{-1} \cdot_{\mathrm{st}} \mathcal{V}\big)\Big).$$

By Lemma 5.3.4.3, the filtered complex map $\mathrm{F}^\bullet_{\mathscr{H}}\, \mathcal{C}^\bullet \twoheadrightarrow \mathrm{F}^\bullet_{\mathscr{H}}\, \mathcal{C}^\bullet_+$ induces a homomorphism of the corresponding convergent spectral sequences which is an isomorphism at the first page:

$$\mathscr{E}^p_1 \cong \mathscr{E}^p_{+,1} \cong \mathrm{H}^p\Big(\mathfrak{n}[t], \mathcal{V}/\big(\mathfrak{m}[t^{-1}]t^{-1} \cdot_{\mathrm{st}} \mathcal{V}\big)\Big).$$

Hence, the infinity pages are isomorphic and we get the cohomology isomorphism that we wanted:

$$\mathrm{H}^p(\mathcal{C}^\bullet, d) \cong \mathrm{H}^p(\mathcal{C}^\bullet_+, d_+).$$

□

5.3.5. Induced Li filtration on BRST cohomology

Let $\rho : \widetilde{X} \to X$ be a Poisson N-equivariant dominant map between two affine Poisson varieties $\widetilde{X}, X$ equipped with algebraic N-actions by Poisson automorphism. We assume that there is a commutative square of Poisson vertex algebra homomorphisms,

$$\begin{tikzcd} \mathrm{gr}^{\mathrm{Li}} \widetilde{V} \arrow[r, "\sim"] & \mathbf{C}[\mathrm{J}_\infty \widetilde{X}] \\ \mathrm{gr}^{\mathrm{Li}} V \arrow[u, hook] \arrow[r, "\sim"'] & \mathbf{C}[\mathrm{J}_\infty X], \arrow[u, hook] \end{tikzcd}$$

where the horizontal arrows are isomorphisms.

Let $\widetilde{\mu} : \widetilde{X} \to \mathfrak{n}^*$ and $\mu : X \to \mathfrak{m}^*$ be two N-equivariant moment maps such that the diagram (5.3.1.1) commutes. One gets the commutative square

$$\begin{tikzcd} \mathbf{C}[\mathrm{J}_\infty \mathfrak{n}^*] \arrow[r, "\mathrm{J}_\infty \widetilde{\mu}^*"] & \mathbf{C}[\mathrm{J}_\infty \widetilde{X}] \\ \mathbf{C}[\mathrm{J}_\infty \mathfrak{m}^*] \arrow[u, hook] \arrow[r, "\mathrm{J}_\infty \mu^*"'] & \mathbf{C}[\mathrm{J}_\infty X], \arrow[u, hook, "\mathrm{J}_\infty \rho^*"'] \end{tikzcd}$$

and we assume that this diagram coincides with the square induced by (5.3.3.1):

$$\begin{tikzcd} \mathrm{gr}^{\mathrm{Li}} \mathcal{V}(\mathfrak{n}) \arrow[r, "\mathrm{gr}^{\mathrm{Li}} \widetilde{\Upsilon}"] & \mathrm{gr}^{\mathrm{Li}} \widetilde{\mathcal{V}} \\ \mathrm{gr}^{\mathrm{Li}} \mathcal{V}(\mathfrak{m}) \arrow[u, hook] \arrow[r, "\mathrm{gr}^{\mathrm{Li}} \Upsilon"'] & \mathrm{gr}^{\mathrm{Li}} \mathcal{V}. \arrow[u, hook] \end{tikzcd}$$

As before, introduce BRST cochain complexes for the vertex algebras,

$$\mathcal{C}^\bullet \coloneqq \mathcal{V} \otimes_{\mathbf{C}} \mathcal{F}^\bullet(\mathfrak{m} \oplus \mathfrak{n}^*) \subseteq \widetilde{\mathcal{C}}^\bullet \coloneqq \widetilde{\mathcal{V}} \otimes_{\mathbf{C}} \mathcal{F}^\bullet(\mathfrak{n} \oplus \mathfrak{n}^*),$$

and the BRST cochain complexes for the arc spaces,

$$C^\bullet_\infty \coloneqq \mathbf{C}[\mathrm{J}_\infty X] \otimes_{\mathbf{C}} \textstyle\bigwedge^\bullet_\infty(\mathfrak{m} \oplus \mathfrak{n}^*) \subseteq \widetilde{C}^\bullet_\infty \coloneqq \mathbf{C}\big[\mathrm{J}_\infty \widetilde{X}\big] \otimes_{\mathbf{C}} \textstyle\bigwedge^\bullet_\infty(\mathfrak{n} \oplus \mathfrak{n}^*).$$

Clearly, one gets the following Poisson vertex superalgebra isomorphisms, compatible with the coboundary operators:

$$\mathrm{gr}^{\mathrm{Li}} \mathcal{C}^\bullet \cong C^\bullet_\infty \quad \text{and} \quad \mathrm{gr}^{\mathrm{Li}} \widetilde{\mathcal{C}}^\bullet \cong \widetilde{C}^\bullet_\infty.$$

The following theorem is a generalisation of [AM25, Theorem 9.7].

Theorem 5.3.5.1. *Consider the data introduced above. Make the following technical assumptions:*

1. *the vertex superalgebra $\widetilde{\mathcal{C}}$ is graded, $\mathcal{C}$ is a graded subalgebra and the element $\widetilde{Q}$ defining the coboundary operator $\widetilde{d}$ is homogeneous of degree 1,*

2. *the space* $\mathcal{I} \coloneqq \mathrm{Span}_{\mathbf{C}}\{\phi_{x(-n-1)}c,\ (d\phi_x)_{(-n-1)}c \mid n \geqslant 0,\ x \in \mathfrak{m},\ c \in \mathcal{C}\}$ *is a graded subspace of* $\mathcal{C}$*,*

3. *the induced grading on the quotient space* $\mathcal{C}_+ \coloneqq \mathcal{C}/\mathcal{I}$ *is nonnegative,*

4. *there is an isomorphism* $\mathrm{H}^\bullet(\mathcal{C}^\bullet, d) \cong \mathrm{H}^\bullet(\mathcal{C}^\bullet_+, d_+)$ *of vector spaces induced by the linear projection* $\mathcal{C}^\bullet \twoheadrightarrow \mathcal{C}^\bullet_+$*,*

5. *the moment map* $\mu : X \to \mathfrak{m}^*$ *is smooth, and there exists a closed subvariety* S *of* $\mu^{-1}(0)$ *such that the action map*
$$\alpha : N \times S \longrightarrow \mu^{-1}(0), \quad (g, x) \longmapsto g \cdot x$$
is an isomorphism.

Then the cohomology vanishes in degrees other than 0*:*
$$\mathrm{H}^n(\mathcal{C}^\bullet, d) = 0 \quad \textit{for} \quad n \neq 0.$$
In degree 0 there is a natural isomorphism
$$\mathrm{gr}^{\mathrm{F}}\, \mathrm{H}^0(\mathcal{C}^\bullet, d) \xrightarrow{\sim} \mathrm{H}^0(\mathrm{gr}^{\mathrm{Li}}\, \mathcal{C}^\bullet, \mathrm{gr}^{\mathrm{Li}}\, d),$$
where the filtration F *on the cohomology* $\mathrm{H}^0(\mathcal{C}^\bullet, d)$ *is induced by the Li filtration on the complex* $(\mathcal{C}^\bullet, d)$*.*

Remark 5.3.5.2. Our assumption (3) is slightly more general than [AM25, Theorem 9.7], where it is assumed also that $\mathcal{C}^\bullet_+(0) = \mathbf{C}\mathbf{1}$. It will be necessary to prove Proposition 6.2.3.1.

Remark 5.3.5.3. In the rest of the paper, we will use Theorem 5.3.4.2 to get the condition (4) of Theorem 5.3.5.1. In particular, this means that two Hamiltonian operators will be needed: we will usually denote by $\mathcal{H}^{\mathrm{old}}$ the one used for Theorem 5.3.4.2 and by $\mathcal{H}^{\mathrm{new}}$ the one used for Theorem 5.3.5.1.

Assume that the hypotheses of Theorem 5.3.5.1 holds. The Li filtration on $\mathcal{C}$ induces a filtration on the quotient $\mathcal{C}_+$, also denoted by F_{Li}. The Li filtration is preserved by the coboundary operators d and d_+ because of the property (5.1.4.1) of this filtration applied to the element $\widetilde{Q}$. Denote by F the filtration induced on their cohomologies. There are natural maps
$$\mathrm{gr}^{\mathrm{F}}\, \mathrm{H}^\bullet(\mathcal{C}^\bullet, d) \longrightarrow \mathrm{H}^\bullet(\mathrm{gr}^{\mathrm{Li}}\, \mathcal{C}^\bullet, \mathrm{gr}^{\mathrm{Li}}\, d),$$
$$\mathrm{gr}^{\mathrm{F}}\, \mathrm{H}^\bullet(\mathcal{C}^\bullet_+, d_+) \longrightarrow \mathrm{H}^\bullet(\mathrm{gr}^{\mathrm{Li}}\, \mathcal{C}^\bullet_+, \mathrm{gr}^{\mathrm{Li}}\, d_+)$$
which make the following square commutative:

(5.3.5.4)
$$\begin{tikzcd}
\mathrm{gr}^{\mathrm{F}}\, \mathrm{H}^\bullet(\mathcal{C}^\bullet, d) \arrow[r] \arrow[d, "\wr"'] & \mathrm{H}^\bullet(\mathrm{gr}^{\mathrm{Li}}\, \mathcal{C}^\bullet, \mathrm{gr}^{\mathrm{Li}}\, d) \arrow[d] \\
\mathrm{gr}^{\mathrm{F}}\, \mathrm{H}^\bullet(\mathcal{C}^\bullet_+, d_+) \arrow[r] & \mathrm{H}^\bullet(\mathrm{gr}^{\mathrm{Li}}\, \mathcal{C}^\bullet_+, \mathrm{gr}^{\mathrm{Li}}\, d_+).
\end{tikzcd}$$

The left vertical map is an isomorphism by assumption of Theorem 5.3.5.1.

Lemma 5.3.5.5. *The natural projection* $\mathrm{gr}^{\mathrm{Li}}\,\mathscr{C}^\bullet \twoheadrightarrow \mathrm{gr}^{\mathrm{Li}}\,\mathscr{C}_+^\bullet$ *induces an isomorphism of vector spaces* $\mathrm{H}^\bullet(\mathrm{gr}^{\mathrm{Li}}\,\mathscr{C}^\bullet, \mathrm{gr}^{\mathrm{Li}}\,\mathscr{d}) \cong \mathrm{H}^\bullet(\mathrm{gr}^{\mathrm{Li}}\,\mathscr{C}_+^\bullet, \mathrm{gr}^{\mathrm{Li}}\,\mathscr{d}_+)$.

Proof. By assumption, there is an isomorphism of Poisson vertex algebras:

$$\mathrm{gr}^{\mathrm{Li}}\,\mathscr{C}^\bullet \cong C_\infty = \mathbf{C}[\mathrm{J}_\infty X] \otimes_{\mathbf{C}} \textstyle\bigwedge_\infty^\bullet(\mathfrak{m} \oplus \mathfrak{n}^*)$$

which maps the boundary operator $\mathscr{d}$ to d_∞. It induces an isomorphism of differential ideals

$$\mathrm{gr}^{\mathrm{Li}}\,\mathscr{I} \cong I_{\mu,\infty} \otimes_{\mathbf{C}} I_{\mathfrak{m},\infty},$$

where $I_{\mu,\infty}$ is the ideal of $\mathbf{C}[\mathrm{J}_\infty X]$ corresponding to the closed embedding of schemes $\mathrm{J}_\infty\mu^{-1}(0) \hookrightarrow \mathrm{J}_\infty X$, and $I_{\mathfrak{m},\infty}$ is the differential ideal of $\bigwedge_\infty^\bullet(\mathfrak{m} \oplus \mathfrak{n}^*)$ spanned by the elements ϕ_x, where x is in $\mathfrak{m}$.

Then we get a cochain complex isomorphism

$$\mathrm{gr}^{\mathrm{Li}}\,\mathscr{C}_+^\bullet \cong \mathbf{C}[\mathrm{J}_\infty\mu^{-1}(0)] \otimes_{\mathbf{C}} \textstyle\bigwedge_\infty^\bullet(\mathfrak{n}^*),$$

where the right-hand side corresponds to the Lie algebra cochain complex of the $\mathfrak{n}[t]$-module $\mathbf{C}[\mathrm{J}_\infty\mu^{-1}(0)]$. Then, it follows from the proof of Theorem 5.3.2.1 that the natural projection $\mathrm{gr}^{\mathrm{Li}}\,\mathscr{C}^\bullet \twoheadrightarrow \mathrm{gr}^{\mathrm{Li}}\,\mathscr{C}_+^\bullet$ induces an isomorphism of vector spaces between their cohomologies. □

For any nonnegative integer m, the Lie algebra cohomology of $\mathbf{C}[\mathrm{J}_m N]$ is the algebraic de Rham cohomology of $\mathrm{J}_m N$. Because this group is unipotent, its cohomology is $\mathbf{C}$ in degree 0 and zero otherwise. Taking the colimit, the Lie algebra cohomology of $\mathbf{C}[\mathrm{J}_\infty N]$ is $\mathbf{C}$ in degree 0, and zero otherwise. Thanks to Theorem 5.3.2.1 and Lemma 5.3.5.5, we deduce that

$$\mathrm{H}^n(\mathrm{gr}^{\mathrm{Li}}\,\mathscr{C}^\bullet, \mathrm{gr}^{\mathrm{Li}}\,\mathscr{d}) = \mathrm{H}^n(\mathrm{gr}^{\mathrm{Li}}\,\mathscr{C}_+^\bullet, \mathrm{gr}^{\mathrm{Li}}\,\mathscr{d}_+) = 0$$

for any nonzero integer n.

Lemma 5.3.5.6. *The cohomology* $\mathrm{H}^n(\mathscr{C}_+^\bullet, \mathscr{d})$ *is zero if the integer n is nonzero, and the natural map* $\mathrm{gr}^{\mathrm{F}}\,\mathrm{H}^\bullet(\mathscr{C}_+^\bullet, \mathscr{d}_+) \to \mathrm{H}^\bullet(\mathrm{gr}^{\mathrm{Li}}\,\mathscr{C}_+^\bullet, \mathrm{gr}^{\mathrm{Li}}\,\mathscr{d}_+)$ *is an isomorphism.*

Proof. Recall that the complex $\mathscr{C}_+$ is nonnegatively graded by the third assumption of Theorem 5.3.5.1. Moreover, by the first assumption, each homogeneous component $\mathscr{C}_+(\Delta)$, for Δ in $\frac{1}{K}\mathbf{Z}_{\geqslant 0}$, is a subcomplex. According to [Ara12, Proposition 2.6.1], $\mathrm{F}_{\mathrm{Li}}^p\,\mathscr{C}_+(\Delta)$ is zero when $p > \Delta$ because the Hamiltonian grading is nonnegative on the quotient complex. So the filtration is finite on each homogeneous component.

To the filtered complex $(\mathrm{F}_{\mathrm{Li}}^\bullet\,\mathscr{C}^\bullet, \mathscr{d}_+)$ is associated a spectral sequence, denoted by $\{(\mathscr{E}_{+,\mathrm{Li},r}, \mathscr{d}_{+,\mathrm{Li},r})\}_{r=0}^\infty$, which is convergent on each homogeneous component for the grading induced by the Hamiltonian operator. The first and infinity pages are:

$$\mathscr{E}_{+,\mathrm{Li},1}^{p,q} = \mathrm{H}^{p+q}(\mathrm{gr}_p^{\mathrm{Li}}\,\mathscr{C}_+^\bullet, \mathrm{gr}^{\mathrm{Li}}\,\mathscr{d}_+),$$
$$\mathscr{E}_{+,\mathrm{Li},\infty}^{p,q} = \mathrm{gr}_p^{\mathrm{F}}\,\mathrm{H}^{p+q}(\mathscr{C}_+^\bullet, \mathscr{d}_+),$$

for p, q in $\mathbf{Z}$. Because of the vanishing implied by the previous lemma and the convergence on homogeneous subcomplexes, we get the desired isomorphism:

$$\mathrm{gr}^{\mathrm{F}} \mathrm{H}^n(\mathcal{C}_+^\bullet, \mathscr{d}_+) \cong \mathrm{H}^n(\mathrm{gr}^{\mathrm{Li}} \mathcal{C}_+^\bullet, \mathrm{gr}^{\mathrm{Li}} \mathscr{d}_+)$$

for any integer n, and both are zero if $n \neq 0$. So we deduce the equality

$$\mathrm{H}^n(\mathcal{C}_+^\bullet, \mathscr{d}) = 0 \quad \text{for} \quad n \neq 0.$$

$\square$

Proof of Theorem 5.3.5.1. We go back to the commutative diagram (5.3.5.4). The vertical maps and the bottom horizontal map are isomorphisms as a consequence of the previous lemmas. So we can conclude that the top map is the desired isomorphism too. The second lemma and the isomorphism

$$\mathrm{H}^\bullet(\mathcal{C}^\bullet, \mathscr{d}) \cong \mathrm{H}^\bullet(\mathcal{C}_+^\bullet, \mathscr{d}_+)$$

given by the fourth condition implies that $\mathrm{H}^n(\mathcal{C}^\bullet, \mathscr{d})$ is zero when n is nonzero. $\square$

6 - Reduction by stages for affine W-algebras

6.1. Construction of affine W-algebras

We construct the affine W-algebra $\mathscr{W}^k(\mathfrak{g}, f)$ by analogy with the finite W-algebra $\mathcal{U}(\mathfrak{g}, f)$. We prove that all the constructions are equivalent to the one due to [KRW03, KW04], by using the equivalence between the constructions of the Slodowy slice S_f (Theorem 6.1.2.4). We recall the Kac–Roan–Wakimoto embedding and give a geometric interpretation to it (Proposition 6.1.4.2).

6.1.1. BRST cohomology

Let $\mathfrak{g}$ be a simple finite-dimensional complex Lie algebra, $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and G be a connected algebraic group whose Lie algebra is $\mathfrak{g}$. Denote by $(\bullet|\bullet) \coloneqq (2h^\vee)^{-1}\kappa_{\mathfrak{g}}$ the normalised non-degenerate symmetric bilinear form on $\mathfrak{g}$.

Let f a nilpotent element of $\mathfrak{g}$ and $\mathfrak{g} = \bigoplus_{\delta\in\mathbf{Z}} \mathfrak{g}_\delta$ be a good grading for f defined by the adjoint action of a semisimple element H in $\mathfrak{h}$. Let $\mathfrak{l}$ be an isotropic subspace of the symplectic space $\mathfrak{g}_1$. Let $\mathfrak{n}_{\mathfrak{l}} \coloneqq \mathfrak{l}^{\perp,\omega} \oplus \mathfrak{g}_{\geqslant 2}$ be a nilpotent subalgebra of $\mathfrak{g}$ and $N_{\mathfrak{l}}$ be the corresponding unipotent subgroup of G.

Denote by $\chi = (f|\bullet)$ the linear form on $\mathfrak{g}$ associated with f and by $\overline{\chi}_{\mathfrak{l}}$ its restriction to $\mathfrak{n}_{\mathfrak{l}}$. Let $\mathbb{O}_{\mathfrak{l}} \coloneqq \mathrm{Ad}^*(N_{\mathfrak{l}})\overline{\chi}_{\mathfrak{l}}$ be its coadjoint orbit. Recall there is a symplectic isomorphism $\sigma_{\mathfrak{l}} : \mathfrak{l}^{\perp,\omega}/\mathfrak{l} \xrightarrow{\sim} \mathbb{O}_{\mathfrak{l}}$.

Consider the tensor product of vertex algebras $\mathscr{V}^k(\mathfrak{g}) \otimes_{\mathbf{C}} \mathscr{A}(\mathfrak{l}^{\perp,\omega}/\mathfrak{l})$. It is equipped with a vertex algebra map

$$\Upsilon_{\mathfrak{l}} : \mathscr{V}(\mathfrak{n}_{\mathfrak{l}}) \longrightarrow \mathscr{V}^k(\mathfrak{g}) \otimes_{\mathbf{C}} \mathscr{A}(\mathfrak{l}^{\perp,\omega}/\mathfrak{l})$$

defined for any element x in $\mathfrak{n}_{\mathfrak{l}}$ by

$$\Upsilon_{\mathfrak{l}}(x) \coloneqq \begin{cases} x + \psi_{(x \bmod \mathfrak{l})} & \text{if} \quad x \in \mathfrak{l}^{\perp,\omega} \\ x + \chi(x)\mathbf{1} & \text{if} \quad x \in \mathfrak{g}_2 \\ x & \text{otherwise.} \end{cases}$$

The data of this chiral moment map allows to define the BRST cochain complex, denoted by $(\mathscr{C}_{\mathfrak{l}}, d_{\mathfrak{l}})$, where

$$\mathscr{C}_{\mathfrak{l}}^\bullet \coloneqq \mathscr{V}^k(\mathfrak{g}) \otimes_{\mathbf{C}} \mathscr{A}(\mathfrak{l}^{\perp,\omega}/\mathfrak{l}) \otimes_{\mathbf{C}} \mathscr{F}^\bullet(\mathfrak{n}_{\mathfrak{l}} \oplus \mathfrak{n}_{\mathfrak{l}}{}^*).$$

Denote by

$$\mathscr{W}^k(\mathfrak{g}, f, H, \mathfrak{l}) \coloneqq \mathrm{H}^\bullet(\mathscr{C}_{\mathfrak{l}}, d_{\mathfrak{l}})$$

the vertex superalgebra constructed by this cohomology.

Remark 6.1.1.1. This is the affine analogue of the BRST complex that is used in [DDCDS+06] to construct finite W-algebra. The W-algebra $\mathscr{W}^k(\mathfrak{g}, f, H, \{0\})$ corresponding to the zero isotropic subspace is the one constructed by Kac, Roan, and Wakimoto in [KRW03]. In [AKM15], an $\hbar$-adic version of the affine W-algebra algebra is constructed for a Lagrangian subspace $\mathfrak{l}$. To our knowledge, the W-algebra $\mathscr{W}^k(\mathfrak{g}, f, H, \mathfrak{l})$ for nonzero $\mathfrak{l}$ appears only in [AM25], for $\mathfrak{l}$ Lagrangian.

Recall from Section 5.2 the following Poisson vertex algebra isomorphisms:

$$\begin{aligned} \operatorname{gr}^{\mathrm{Li}} \mathcal{V}^k(\mathfrak{g}) &\cong \mathbf{C}[\mathrm{J}_\infty \mathfrak{g}^*], \quad \operatorname{gr}^{\mathrm{Li}} \mathcal{A}(\mathfrak{l}^{\perp,\omega}/\mathfrak{l}) \cong \mathbf{C}[\mathfrak{l}^{\perp,\omega}/\mathfrak{l}] \\ \operatorname{gr}^{\mathrm{Li}} \mathcal{V}(\mathfrak{n}_\mathfrak{l}) &\cong \mathbf{C}[\mathrm{J}_\infty {\mathfrak{n}_l}^*], \quad \operatorname{gr}^{\mathrm{Li}} \mathcal{F}^\bullet(\mathfrak{n}_\mathfrak{l} \oplus {\mathfrak{n}_\mathfrak{l}}^*) \cong \textstyle\bigwedge_\infty^\bullet(\mathfrak{n}_\mathfrak{l} \oplus {\mathfrak{n}_\mathfrak{l}}^*). \end{aligned}$$

In Section 2.2.3, we introduced the moment map

$$\mu_\mathfrak{l} : \mathfrak{g}^* \times (\mathfrak{l}^{\perp,\omega}/\mathfrak{l}) \longrightarrow {\mathfrak{n}_\mathfrak{l}}^*, \quad (\xi, v \bmod \mathfrak{l}) \longmapsto \pi_\mathfrak{l}(\xi) + \overline{\chi}_\mathfrak{l} + \operatorname{ad}^*(v)\overline{\chi}_\mathfrak{l},$$

where $\pi_\mathfrak{l} : \mathfrak{g}^* \twoheadrightarrow {\mathfrak{n}_\mathfrak{l}}^*$ is the restriction map. If one takes the graded object corresponding to the Li filtration on the homomorphism $\Upsilon_\mathfrak{l}$ of vertex algebras, one gets the arcs of the moment map $\mu_\mathfrak{l}$. Denote by $(C^\bullet_{\mathfrak{l},\infty}, d_{\mathfrak{l},\infty})$ the associated Poisson vertex BRST complex, where:

$$C^\bullet_{\mathfrak{l},\infty} := \mathbf{C}[\mathrm{J}_\infty \mathfrak{g}^*] \otimes_\mathbf{C} \mathbf{C}[\mathrm{J}_\infty(\mathfrak{l}^{\perp,\omega}/\mathfrak{l})] \otimes_\mathbf{C} \textstyle\bigwedge_\infty^\bullet(\mathfrak{n}_\mathfrak{l} \oplus {\mathfrak{n}_\mathfrak{l}}^*).$$

There is a Poisson vertex isomorphism $\operatorname{gr}^{\mathrm{Li}} \mathcal{C}^\bullet_\mathfrak{l} \cong C^\bullet_{\mathfrak{l},\infty}$ that respects the cochain structure.

The following theorem generalises [KW04, Theorem 4.1] for $\mathfrak{l} = \{0\}$ and [AM25, Theorem 9.7] for $\mathfrak{l}$ being Lagrangian.

Theorem 6.1.1.2. *The following cohomology vanishes outside degree* 0*:*

$$\mathrm{H}^n(\mathcal{C}_\mathfrak{l}, d_\mathfrak{l}) \quad \textit{for} \quad n \neq 0,$$

and there is a natural isomorphism

$$\operatorname{gr}^{\mathrm{F}} \mathrm{H}^0(\mathcal{C}^\bullet_\mathfrak{l}, d_\mathfrak{l}) \xrightarrow{\sim} \mathrm{H}^0(C^\bullet_{\mathfrak{l},\infty}, d_{\mathfrak{l},\infty})$$

where the filtration F *on the cohomology* $\mathrm{H}^0(\mathcal{C}_\mathfrak{l}, d_\mathfrak{l})$ *is induced by the Li filtration on the complex* $(\mathcal{C}_\mathfrak{l}, d_\mathfrak{l})$*. In particular,* $\mathscr{W}^k(\mathfrak{g}, f, H, \mathfrak{l}) = \mathrm{H}^0(\mathcal{C}_\mathfrak{l}, d_\mathfrak{l})$ *is a purely even vertex algebra.*

It is a straightforward generalisation [AM25, Theorem 9.7], we only sketch the proof.

Sketch of proof. We first apply Theorem 5.3.4.2 to the BRST complex $(\mathcal{C}_\mathfrak{l}, d_\mathfrak{l})$ where the vertex algebra $\mathcal{V}^k(\mathfrak{g}) \otimes_\mathbf{C} \mathcal{A}(\mathfrak{l}^{\perp,\omega}/\mathfrak{l})$ is graded by the Hamiltonian operator given by $\mathcal{H}^{\mathrm{old}} := \mathcal{H}^\mathfrak{g} + L_0^{\mathcal{A}}$. So, we get an isomorphism

$$\mathrm{H}^n(\mathcal{C}^\bullet_\mathfrak{l}, d_\mathfrak{l}) \cong \mathrm{H}^n(\mathcal{C}^\bullet_{\mathfrak{l},+}, d_{\mathfrak{l},+})$$

where $(\mathcal{C}^\bullet_{\mathfrak{l},+}, d_{\mathfrak{l},+})$ is the quotient complex defined as in Section 5.3.4. See [AM25, Proposition 9.3] for details.

Fix a basis $\{x_i\}_{i\in I(\mathfrak{n}_\mathfrak{l})}$ of $\mathfrak{n}_\mathfrak{l}$ which is homogeneous for the good grading and denote by $\delta(x_i)$ the degree of x_i. Equip the complex $\mathcal{C}_\mathfrak{l}$ with the Hamiltonian operator

$$\mathcal{H}^{\mathrm{new}} := \mathcal{H}^{\mathfrak{g}} + \frac{1}{2}(\partial H)_{(1)} + L_0^{\mathcal{A}} + L^{\mathcal{F}}(m_\bullet)_0, \quad \text{where} \quad m_\bullet = \frac{1}{2}\delta(x_\bullet).$$

Thanks to the smoothness of $\mu_\mathfrak{l}$ (it is a surjective affine map) and of the isomorphism $N_\mathfrak{l} \times S_f \cong {\mu_\mathfrak{l}}^{-1}(0)$ stated in Theorem 2.2.3.3, we can apply Theorem 5.3.5.1 and conclude the proof. □

6.1.2. Equivalence of the constructions

In the previous section, all the objects related to $\mathfrak{l}$ were denoted with a "$\mathfrak{l}$" subscript. For $\mathfrak{l} = \{0\}$, we omit the subscript:

$$\mathfrak{n} = \mathfrak{g}_{\geqslant 1}, \quad N = G_{\geqslant 1}, \quad \overline{\chi} = \chi|_{\mathfrak{n}}, \quad \mathbb{O} = \mathrm{Ad}^*(N)\overline{\chi}, \quad \sigma : \mathfrak{g}_1 \xrightarrow{\sim} \mathbb{O}.$$

There is the twisted moment map

$$\mu : \mathfrak{g}^* \times \mathfrak{g}_1 \longrightarrow \mathfrak{n}^*, \quad (\xi, v) \longmapsto \pi(\xi) + \overline{\chi} + \mathrm{ad}^*(v)\overline{\chi},$$

where $\pi : \mathfrak{g}^* \twoheadrightarrow \mathfrak{n}^*$ denotes the restriction map.

The vertex algebra BRST complex is denoted by $(\mathcal{C}^\bullet, d)$, with

$$\mathcal{C}^\bullet = \mathcal{V}^k(\mathfrak{g}) \otimes_{\mathbf{C}} \mathcal{A}(\mathfrak{g}_1) \otimes_{\mathbf{C}} \mathcal{F}^\bullet(\mathfrak{n} \oplus \mathfrak{n}^*).$$

We want to compare its cohomology to the one of the BRST complex for a generic isotropic subspace $\mathfrak{l}$ of $\mathfrak{g}_1$, denoted by $(\mathcal{C}^\bullet_\mathfrak{l}, d_\mathfrak{l})$.

Denote by $\mathcal{A}(\mathfrak{l}^{\perp,\omega})$ the vertex subalgebra of the Weyl vertex algebra $\mathcal{A}(\mathfrak{g}_1)$ which strongly generated by the ψ_v for v in $\mathfrak{l}^{\perp,\omega}$. The map

$$\Upsilon : \mathcal{V}(\mathfrak{g}_{\geqslant 1}) \longrightarrow \mathcal{V}^k(\mathfrak{g}) \otimes_{\mathbf{C}} \mathcal{A}(\mathfrak{g}_1)$$

restricts to a vertex algebra homomorphism

$$\Upsilon_{\mathrm{int}} : \mathcal{V}(\mathfrak{n}_\mathfrak{l}) \longrightarrow \mathcal{V}^k(\mathfrak{g}) \otimes_{\mathbf{C}} \mathcal{A}(\mathfrak{l}^{\perp,\omega}).$$

As in [AKM15, Paragraph 3.2.5], we can define an *intermediary BRST complex* denoted by $(\mathcal{C}^\bullet_{\mathrm{int}}, d_{\mathrm{int}})$ where $\mathcal{C}_{\mathrm{int}}$ is the subalgebra of $\mathcal{C}$ defined by

$$\mathcal{C}^\bullet_{\mathrm{int}} := \mathcal{V}^k(\mathfrak{g}) \otimes_{\mathbf{C}} \mathcal{A}(\mathfrak{l}^{\perp,\omega}) \otimes_{\mathbf{C}} \mathcal{F}^\bullet(\mathfrak{n}_\mathfrak{l} \oplus \mathfrak{n}^*),$$

and the operator d_{int} is the restriction of the coboundary operator d to the subalgebra $\mathcal{C}_{\mathrm{int}}$.

By construction, one has the injective homomorphism

$$\Pi_1 : (\mathcal{C}^\bullet_{\mathrm{int}}, d_{\mathrm{int}}) \hookrightarrow (\mathcal{C}^\bullet, d)$$

of cochain complexes, and compatible with the vertex superalgebra structure. The linear projections $\mathfrak{l}^{\perp,\omega} \twoheadrightarrow \mathfrak{l}^{\perp,\omega}/\mathfrak{l}$ and $\mathfrak{n}^* \twoheadrightarrow {\mathfrak{n}_\mathfrak{l}}^*$ induce a surjective homomorphism

$$\Pi_2 : (\mathcal{C}^\bullet_{\mathrm{int}}, d_{\mathrm{int}}) \longrightarrow\!\!\!\!\!\rightarrow (\mathcal{C}^\bullet_{\mathfrak{l}}, d_{\mathfrak{l}})$$

of cochain complexes and vertex superalgebras.

Let us denote by

$$\mathbb{O}_{\mathrm{int}} \coloneqq \mathrm{Ad}^*(N)\overline{\chi}_\mathfrak{l}$$

the $N_\mathfrak{l}$-orbit in ${\mathfrak{n}_\mathfrak{l}}^*$ of the restriction of χ to $\mathfrak{n}_\mathfrak{l}$, called the *intermediary orbit*. The following lemma is proved in the same way as Proposition 6.2.2.2.

Lemma 6.1.2.1. *The isomorphism $\sigma : \mathfrak{g}_1 \xrightarrow{\sim} \mathbb{O}_{\max}$ induces an isomorphism*

$$\sigma_{\mathrm{int}} : \mathfrak{g}_1/\mathfrak{l} \xrightarrow{\sim} \mathbb{O}_{\mathrm{int}}, \quad (v \bmod \mathfrak{l}) \longmapsto \overline{\chi}_\mathfrak{l} + \mathrm{ad}^*(v)\overline{\chi}_\mathfrak{l}.$$

On the coordinate ring side, we see that $\mathbf{C}[\mathfrak{g}_1/\mathfrak{l}] = \mathrm{Sym}\,\mathfrak{l}^{\perp,\omega}$. The latter is a Poisson subalgebra of $\mathrm{Sym}\,\mathfrak{g}_1$, whence $\mathfrak{g}_1/\mathfrak{l} \cong \mathbb{O}_{\mathrm{int}}$ is a Poisson variety.

Remark 6.1.2.2. In [AKM15], they do not need an intermediary orbit because their choice of intermediate complex is slightly different from ours. We modify their construction in the perspective of proving Theorem 5 (6.2.3.7), for which such intermediary orbit is necessary.

The intermediary moment map is defined as

$$\mu_{\mathrm{int}} : \mathfrak{g}^* \times (\mathfrak{g}_1/\mathfrak{l}) \longrightarrow {\mathfrak{n}_\mathfrak{l}}^*, \quad (\xi, v) \longmapsto \pi_\mathfrak{l}(\xi) + \overline{\chi}_\mathfrak{l} + \mathrm{ad}^*(v)\overline{\chi}_\mathfrak{l}.$$

It is an N-equivariant homomorphism because $N_\mathfrak{l}$ is a normal subgroup of N.

Lemma 6.1.2.3. *The projection map and the embedding*

$$\mathfrak{g}^* \times \mathfrak{g}_1 \longrightarrow\!\!\!\!\!\rightarrow \mathfrak{g}^* \times (\mathfrak{g}_1/\mathfrak{l}) \quad \textit{and} \quad \mathfrak{g}^* \times (\mathfrak{l}^{\perp,\omega}/\mathfrak{l}) \hookrightarrow \mathfrak{g}^* \times (\mathfrak{g}_1/\mathfrak{l})$$

induce Poisson isomorphisms between the corresponding Hamiltonian reductions:

$${\mu_\mathfrak{l}}^{-1}(0)/\!/N_\mathfrak{l} \cong {\mu_{\mathrm{int}}}^{-1}(0)/\!/N \cong \mu^{-1}(0)/\!/N.$$

Proof. The projection maps $\mathfrak{g}^* \times \mathfrak{g}_1 \twoheadrightarrow \mathfrak{g}^* \times (\mathfrak{g}_1/\mathfrak{l}) \twoheadrightarrow \mathfrak{g}^*$ induces an N-equivariant isomorphism

$$\mu^{-1}(0) \cong {\mu_{\mathrm{int}}}^{-1}(0) \cong -\chi + (\mathfrak{g}_{\geqslant 2})^\perp.$$

It is clear because of the parametrization σ_{int} of the intermediary orbit $\mathbb{O}_{\mathrm{int}}$. Then the lemma follows from Corollary 2.2.5.1. $\square$

Theorem 6.1.2.4. *The following inclusion and projection maps of cochain complexes*

$$\Pi_1 : (\mathcal{C}^\bullet_{\mathrm{int}}, d_{\mathrm{int}}) \hookrightarrow (\mathcal{C}^\bullet, d) \quad \textit{and} \quad \Pi_2 : (\mathcal{C}^\bullet_{\mathrm{int}}, d_{\mathrm{int}}) \longrightarrow\!\!\!\!\!\rightarrow (\mathcal{C}^\bullet_{\mathfrak{l}}, d_{\mathfrak{l}})$$

induce vertex algebra isomorphisms between their cohomologies:

$$\mathrm{H}^0(\mathcal{C}^\bullet_{\mathfrak{l}}, d_{\mathfrak{l}}) \cong \mathrm{H}^0(\mathcal{C}^\bullet_{\mathrm{int}}, d_{\mathrm{int}}) \cong \mathrm{H}^0(\mathcal{C}^\bullet, d).$$

Proof. Thanks to Lemma 6.1.2.3, we can apply Theorems 5.3.4.2 and 5.3.5.1 to the cochain complex $(\mathscr{C}^\bullet_{\mathrm{int}}, \mathscr{d}_{\mathrm{int}})$ seen as a subcomplex of $(\mathscr{C}^\bullet, \mathscr{d})$ and we get:

$$\mathrm{H}^n(\mathscr{C}^\bullet_{\mathrm{int}}, \mathscr{d}_{\mathrm{int}}) = 0 \quad \text{for} \quad n \neq 0,$$
$$\mathrm{gr}^{\mathrm{F}} \mathrm{H}^0(\mathscr{C}^\bullet_{\mathrm{int}}, \mathscr{d}_{\mathrm{int}}) \cong \mathrm{H}^0(\mathrm{gr}^{\mathrm{Li}} \mathscr{C}^\bullet_{\mathrm{int}}, \mathrm{gr}^{\mathrm{Li}} \mathscr{d}_{\mathrm{int}}).$$

The cochain maps Π_1 and Π_2 induce vertex algebra maps after taking the cohomology, denoted by

$$\Psi_1 : \mathrm{H}^0(\mathscr{C}^\bullet_{\mathrm{int}}, \mathscr{d}_{\mathrm{int}}) \longrightarrow \mathrm{H}^0(\mathscr{C}^\bullet, \mathscr{d}) \quad \text{and} \quad \Psi_2 : \mathrm{H}^0(\mathscr{C}^\bullet_{\mathrm{int}}, \mathscr{d}_{\mathrm{int}}) \longrightarrow \mathrm{H}^0(\mathscr{C}^\bullet_{\mathfrak{l}}, \mathscr{d}_{\mathfrak{l}}).$$

Consider the Poisson vartiety isomorphisms of Lemma 6.1.2.3. They induce isomorphisms of Poisson vertex algebras after passing to arc spaces and coordinate rings. These isomorphisms coincide with the associated graded maps

$$\mathrm{gr}^{\mathrm{F}} \Psi_1 : \mathrm{H}^0(\mathrm{gr}^{\mathrm{Li}} \mathscr{C}^\bullet_{\mathrm{int}}, \mathrm{gr}^{\mathrm{Li}} \mathscr{d}_{\mathrm{int}}) \longrightarrow \mathrm{H}^0(\mathrm{gr}^{\mathrm{Li}} \mathscr{C}^\bullet, \mathrm{gr}^{\mathrm{Li}} \mathscr{d})$$

and

$$\mathrm{gr}^{\mathrm{F}} \Psi_2 : \mathrm{H}^0(\mathrm{gr}^{\mathrm{Li}} \mathscr{C}^\bullet_{\mathrm{int}}, \mathrm{gr}^{\mathrm{Li}} \mathscr{d}_{\mathrm{int}}) \longrightarrow \mathrm{H}^0(\mathrm{gr}^{\mathrm{Li}} \mathscr{C}^\bullet_{\mathfrak{l}}, \mathrm{gr}^{\mathrm{Li}} \mathscr{d}_{\mathfrak{l}}),$$

hence $\mathrm{gr}^{\mathrm{F}} \Psi_1$ and $\mathrm{gr}^{\mathrm{F}} \Psi_2$ are isomorphisms.

The Hamiltonian operator $\mathscr{H}^{\mathrm{new}}$ introduced in the proof of Theorem 6.1.1.2 induces a nonnegative grading on the three vertex algebras

$$\mathrm{H}^0(\mathscr{C}^\bullet_{\mathfrak{l}}, \mathscr{d}_{\mathfrak{l}}), \quad \mathrm{H}^0(\mathscr{C}^\bullet_{\mathrm{int}}, \mathscr{d}_{\mathrm{int}}) \quad \text{and} \quad \mathrm{H}^0(\mathscr{C}^\bullet, \mathscr{d}), \tag{6.1.2.5}$$

and the maps Ψ_1 and Ψ_2 commute with the Hamiltonian operators. The filtrations F induced on the cohomologies by the Li filtrations on the BRST complexes are finite on each homogeneous component of these Hamiltonian gradings. This follows from the fact that the cohomologies (6.1.2.5) are isomorphic to

$$\mathrm{H}^0(\mathscr{C}^\bullet_{\mathfrak{l},+}, \mathscr{d}_{\mathfrak{l},+}), \quad \mathrm{H}^0(\mathscr{C}^\bullet_{\mathrm{int},+}, \mathscr{d}_{\mathrm{int},+}) \quad \text{and} \quad \mathrm{H}^0(\mathscr{C}^\bullet_{+}, \mathscr{d}_{+}),$$

and it follows from the proof of Lemma 5.3.5.6 that the filtrations F on the homogeneous components of these cohomologies are finite.

Because the maps $\mathrm{gr}^{\mathrm{F}} \Psi_1$ and $\mathrm{gr}^{\mathrm{F}} \Psi_1$ are isomorphisms, and because the filtrations are finite, the maps Ψ_1 and Ψ_2 restrict to isomorphisms between the homogeneous components, so they are indeed isomorphisms. □

Denote by $\mathscr{W}^k(\mathfrak{g}, f, H)$ the unique vertex algebra obtained by computing the cohomology $\mathrm{H}^0(\mathscr{C}^\bullet_{\mathfrak{l}}, \mathscr{d}_{\mathfrak{l}})$ for any isotropic subspace $\mathfrak{l}$ of $\mathfrak{g}_1$. It is called the *affine W-algebra* associated with the good pair (f, H) at level k.

Theorem 6.1.2.6. *For any nilpotent elements f, f' in the orbit* $\mathbf{O}$ *and for any semisimple elements H, H' defining good gradings for these nilpotent elements, there is a vertex algebra isomorphism between the associated W-algebras:*

$$\mathscr{W}^k(\mathfrak{g}, f, H) \cong \mathscr{W}^k(\mathfrak{g}, f', H').$$

Proof. With Theorem 6.1.2.4, the arguments used in [BG07, Sections 4 and 5] to prove the analogue statement for finite W-algebras by the way of adjacent good gradings can be applied to affine W-algebras. □

As a consequence, the affine W-algebras $\mathscr{W}^k(\mathfrak{g}, f, H)$ constructed above only depend on the nilpotent orbit $\mathbf{O}$ which contains f. They are called *the affine W-algebra associated with the orbit* $\mathbf{O}$ *at level* k and denoted by $\mathscr{W}^k(\mathfrak{g}, f)$.

6.1.3. About the structure of the affine W-algebra

Let us consider the construction of the W-algebra for the trivial isotropic subspace $\mathfrak{l} = \{0\}$:

$$\mathscr{W}^k(\mathfrak{g}, f) = \mathrm{H}^0(\mathscr{C}^\bullet, \mathscr{d}), \quad \text{where} \quad \mathscr{C}^\bullet := \mathscr{V}^k(\mathfrak{g}) \otimes_{\mathbf{C}} \mathscr{A}(\mathfrak{g}_1) \otimes_{\mathbf{C}} \mathscr{F}^\bullet(\mathfrak{n} \oplus \mathfrak{n}^*)$$

and $\mathfrak{n}$ stands for the nilpotent Lie algebra $\mathfrak{g}_{\geqslant 1}$. This is the construction originally done by Kac, Roan and Wakimoto in [KRW03, KW04].

Let us denote by $\{x_i\}_{i\in I(\mathfrak{g})}$ a basis of the Lie algebra $\mathfrak{g}$ containing a basis $\{x_i\}_{i\in I(\mathfrak{n})}$ of the subagebra $\mathfrak{n}$ and assume that these bases are homogeneous the good grading of $\mathfrak{g}$. Denote by $\delta(x_i)$ the degree of x_i. For any element x in the Lie algebra $\mathfrak{g}$, introduce the element

$$J^x := x + \sum_{j,k\in I(\mathfrak{n})} c_j^k(x){:}\phi_k \phi_j^*{:} \quad \text{of} \quad \mathscr{C}_0,$$

where the $c_j^k(x)$ are the structure constants defined by the relations

$$[x, x_j] = \sum_{k\in I(\mathfrak{g})} c_j^k(x) x_k. \tag{6.1.3.1}$$

The BRST complex $\mathscr{C}^\bullet$ contains two interesting vertex subalgebras. The first one, denoted by $\mathscr{C}_+$, is strongly generated by J^x for x in the negative parabolic subalgebra $\mathfrak{g}_{\leqslant 0}$, the subalgebra $\mathscr{A}(\mathfrak{g}_1)$ and the positively charged generators ϕ_ξ^* for ξ in $\mathfrak{n}^*$. The second one, denoted by $\mathscr{C}_-$, is strongly generated by the negatively charged generators ϕ_x for x in $\mathfrak{n}$ and their images $\mathscr{d}(\phi_x)$ by the coboundary operator.

Both are graded by the charge and stable by $\mathscr{d}$. There is the tensor product decomposition $\mathscr{C}^\bullet = \mathscr{C}_+^\bullet \otimes_{\mathbf{C}} \mathscr{C}_-^\bullet$. Note that the subcomplex $\mathscr{C}_+^\bullet$ is nonnegatively graded for the charge, so $\mathrm{H}^0(\mathscr{C}_+^\bullet, \mathscr{d}) = \mathrm{Ker}(\mathscr{d}) \cap \mathscr{C}_+^0$.

Proposition 6.1.3.2 ([KW04, (4.5) and Theorem 4.1])**.** *The vertex algebra inclusion* $\mathscr{C}_+^\bullet \hookrightarrow \mathscr{C}^\bullet$ *induces an isomorphism between the cohomologies:*

$$\mathrm{Ker}(\mathscr{d}) \cap \mathscr{C}_+^0 \cong \mathscr{W}^k(\mathfrak{g}, f).$$

We equip the complex $\mathscr{C}$ with the Hamiltonian operator

$$\mathscr{H} := \mathscr{H}^{\mathfrak{g}} - \frac{1}{2}(\partial H)_{(1)} + L_0^{\mathscr{A}} + L^{\mathscr{F}}(m_\bullet)_0, \quad \text{where} \quad m_\bullet = \frac{1}{2}\delta(x_\bullet). \tag{6.1.3.3}$$

It commutes with the coboundary operator $\mathscr{d}$ and then it induces a grading on the W-algebra $\mathscr{W}^k(\mathfrak{g}, f)$. If the element x in $\mathfrak{g}$ is homogeneous of degree $\delta(x)$ for the good grading, then the conformal degree of J^x is $\Delta(J^x) = 1 - \frac{1}{2}\delta(x)$.

Theorem 6.1.3.4 ([KW04, Theorem 4.1])**.** *Let $\{x_i\}_{i=1}^{\ell}$ be a basis of $\mathfrak{g}^f$ which is homogeneous for the good grading. For each index $1 \leqslant i \leqslant \ell$, one can construct an element $J^{\{i\}}$ in $\mathrm{Ker}(\mathscr{d}) \cap C_+^0 \cong \mathscr{W}^k(\mathfrak{g}, f)$ such that the following properties hold.*

1. *The family $\{J^{\{i\}}\}_{i=1}^{\ell}$ freely generates the W-algebra.*
2. *The Hamiltonian degree of each $J^{\{i\}}$ is $1 - \frac{1}{2}\delta(x_i)$.*
3. *The elements $J^{\{i\}}$ are of the form $J^{\{i\}} = J^{x_i} + T_i$, where T_i is a linear combination of normally ordered products of elements of the form*

$$\partial^n(J^{x_j}) \quad \textit{where} \quad n \in \mathbf{Z}_{\geqslant 0} \quad \textit{and} \quad \delta(x_i) < \delta(x_j) \leqslant 0,$$

or of the form

$$\partial^n(\psi_v) \quad \textit{where} \quad n \in \mathbf{Z}_{\geqslant 0} \quad \textit{and} \quad v \in \mathfrak{g}_1,$$

so that the Hamiltoniam degree of each product equals $1 - \frac{1}{2}\delta(x_i)$.

6.1.4. Geometric interpretation of the Kac–Roan–Wakimoto embedding

Recall that $G^\natural$ is the connected subgroup of G whose Lie algebra is

$$\mathfrak{g}^\natural := \mathfrak{g}^f \cap \mathfrak{g}_0.$$

Let $\{v_i\}_{i=1}^{2s}$ be a basis of the symplectic space $\mathfrak{g}_1$ and $\{v^i\}_{i=1}^{2s}$ be the symplectic dual basis. These vectors correspond to fields ψ_i and ψ^i in the Weyl vertex algebra $\mathscr{A}(\mathfrak{g}_1)$. We state the Kac–Roan–Wakimoto embedding.

Proposition 6.1.4.1 ([KRW03, Theorem 2.4])**.** *For any x in $\mathfrak{g}^\natural$, the element of C_+^0 defined by the formula*

$$J^{\{x\}} := J^x + \frac{1}{2}\sum_{i=1}^{2s} {:}\psi^i \psi_{[u_i,x]}{:}$$

belongs to the W-algebra $\mathscr{W}^k(\mathfrak{g}, f)$.

Moreover, there is an invariant symmetric bilinear form τ_k on the Lie algebra $\mathfrak{g}^\natural$ such that the mapping $x \mapsto J^{\{x\}}$ induces an injective vertex algebra embedding

$$\Theta^\natural : \mathscr{V}^{\tau_k}(\mathfrak{g}^\natural) \longrightarrow \mathscr{W}^k(\mathfrak{g}, f).$$

In [KRW03, KW04], the formula defining $\Theta^\natural$ comes from the explicit computation of the low degree generators in Theorem 6.1.3.4. This formula has in fact a very natural geometric interpretation. In Section 2.2.6, we used reduction by stages to reprove a well-known result of Premet: $G^\natural$ acts on $\mu^{-1}(0)//N$ with a moment map $\theta^\natural : \mu^{-1}(0)//N \to (\mathfrak{g}^\natural)^*$ whose comorphism is given for x in $\mathfrak{g}^\natural$ by:

$$(\theta^\natural)^*(x) = x \otimes 1 + \frac{1}{2}\sum_{i=1}^{2s} 1 \otimes v^i[v_i, x] \bmod I,$$

that lies in $\big((\mathbf{C}[\mathfrak{g}^*] \otimes_{\mathbf{C}} \mathbf{C}[\mathfrak{g}_1])/I\big)^N$. The following proposition is then clear.

Proposition 6.1.4.2. *The moment map $\theta^\natural : \mu^{-1}(0)//N \to (\mathfrak{g}^\natural)^*$ is the geometric analogue of the vertex algebra embedding $\Theta^\natural : \mathcal{V}^{\tau_k}(\mathfrak{g}^\natural) \hookrightarrow \mathcal{W}^k(\mathfrak{g}, f)$ in the sense that the following square of Poisson vertex algebra homomorphism commutes:*

$$\begin{tikzcd}
\operatorname{gr}^{\mathrm{Li}} \mathcal{V}^{\tau_k}(\mathfrak{g}^\natural) \arrow[r, "\sim"] \arrow[d, "\operatorname{gr}^{\mathrm{Li}} \Theta^\natural"'] & \mathbf{C}[\mathrm{J}_\infty(\mathfrak{g}^\natural)^*] \arrow[d, "(\mathrm{J}_\infty \theta^\natural)^*"] \\
\operatorname{gr}^{\mathrm{Li}} \mathcal{W}^k(\mathfrak{g}, f) \arrow[r, "\sim"'] & \mathbf{C}[\mathrm{J}_\infty(\mu^{-1}(0)//N)].
\end{tikzcd}$$

The horizontal isomorphisms are provided by Section 5.2.1 and Theorem 6.1.1.2.

6.2. Chiral reduction by stages

We state reduction by stages for affine W-algebras under Conditions ($\bigstar$), see Theorem 4 (6.2.1.3). To prove it, we provide a new construction of the Slodowy slice S_2 (Theorem 6.2.2.6) to deduce a new construction of $\mathcal{W}^k(\mathfrak{g}, f_2)$, that is to say Theorem 5 (6.2.3.7).

6.2.1. Reduction by stages for affine W-algebras

For $i = 1, 2$, let f_i be a nilpotent element and H_i be in the Cartan subalgebra $\mathfrak{h}$. Assume that the Lie algebra grading $\mathfrak{g} = \bigoplus_{\delta \in \mathbf{Z}} \mathfrak{g}_\delta^{(i)}$ defined by the adjoint action of H_i is a good grading for f_i. Since $[H_1, H_2] = 0$, the good gradings define a bigrading $\mathfrak{g} = \bigoplus_{\delta_1, \delta_2 \in \mathbf{Z}} \mathfrak{g}_{\delta_1, \delta_2}$. Set $f_0 \coloneqq f_2 - f_1$.

Assume that Conditions ($\bigstar$) hold and consider the nilpotent subalgebras $\mathfrak{n}_2$, $\mathfrak{n}_1$ and $\mathfrak{n}_0$ constructed in Section 2.3.1. Recall that the corresponding unipotent subgroups of the reductive group G satisfy the semi-direct product decomposition $N_2 = N_1 \rtimes N_0$. For $i = 1, 2$, one has by definition

$$\mathfrak{n}_i = \mathfrak{l}_i^{\perp, \omega_i} \oplus \mathfrak{g}_{\geqslant 2}^{(i)},$$

where $\mathfrak{l}_i$ is an isotropic subspace of $\mathfrak{g}_1^{(i)}$ such that the symplectic isomorphism $\mathfrak{g}_{1,1} \cong \mathfrak{l}_i^{\perp, \omega_i}/\mathfrak{l}_i$ holds.

Introduce the nilpotent Lie algebras

$$\widetilde{\mathfrak{n}}_1 \coloneqq \mathfrak{g}_{\geqslant 1}^{(1)}.$$

The Lie algebra $\widetilde{\mathfrak{n}}_1$ corresponds to the choice of the zero isotropic subspace of $\mathfrak{g}_1^{(1)}$ in the construction of the affine W-algebra $\mathscr{W}^k(\mathfrak{g}, f_1)$ described in Subsection 6.1.1:

$$\mathscr{W}^k(\mathfrak{g}, f_1) = \mathrm{H}^0\big(\widetilde{\mathscr{C}}_1^\bullet, \widetilde{d}_1\big), \quad \text{where} \quad \widetilde{\mathscr{C}}_1^\bullet \coloneqq \mathscr{V}^k(\mathfrak{g}) \otimes_{\mathbf{C}} \mathscr{A}(\mathfrak{g}_1^{(1)}) \otimes_{\mathbf{C}} \mathscr{F}^\bullet(\widetilde{\mathfrak{n}}_1 \oplus \widetilde{\mathfrak{n}}_1{}^*).$$

Moreover, as recalled in Proposition 6.1.3.2, the affine W-algebra can be constructed as a vertex subalgebra of the BRST complex:

$$\mathscr{W}^k(\mathfrak{g}, f_1) \subseteq \mathrm{Ker}(d_1) \cap \widetilde{\mathscr{C}}_1^0. \tag{6.2.1.1}$$

Let $\{v_i\}_{i=1}^{2s}$ be a basis of the symplectic space $\mathfrak{g}_1^{(1)}$ and denote by $\{v^i\}_{i=1}^{2s}$ the symplectic dual basis. These vectors correspond to fields ψ_i and ψ^i in the Weyl vertex algebra $\mathscr{A}(\mathfrak{g}_1^{(1)})$. Denote by $\{x_i\}_{i\in I(\widetilde{\mathfrak{n}}_1)}$ a basis of the Lie algebra $\widetilde{\mathfrak{n}}_1$ and by ϕ_i, ϕ_i^* the corresponding strong generators in the Clifford vertex algebra denoted by $\mathscr{F}^\bullet(\widetilde{\mathfrak{n}}_1 \oplus \widetilde{\mathfrak{n}}_1{}^*)$. Denote by $c_j^k(x)$ the structure coefficient as defined in the formula (6.1.3.1).

Lemma 6.2.1.2. *There is an embedding of vertex algebras $\mathscr{V}(\mathfrak{n}_0) \hookrightarrow \mathscr{W}^k(\mathfrak{g}, f_1)$ given for any x in $\mathfrak{n}_0$ by the formula*

$$x \longmapsto J^{\{x\}} = x + \frac{1}{2}\sum_{i=1}^{2s} {:}\psi^i \psi_{[v_i,x]}{:} + \sum_{j,k\in I(\widetilde{\mathfrak{n}}_1)} c_j^k(x) {:}\phi_k \phi_j^*{:},$$

where the right-hand side element is a priori *defined in $\widetilde{\mathscr{C}}_1^0$.*

Proof. According to Proposition 6.1.4.1, there exists an invariant symmetric bilinear form $\tau_k^{(1)}$ on the Lie algebra $\mathfrak{g}^{\natural,1} = \mathfrak{g}_0^{(1)} \cap \mathfrak{g}^{f_1}$ such that there is an explicit vertex algebra embedding $\mathscr{V}^{\tau_k^{(1)}}(\mathfrak{g}^{\natural,1}) \hookrightarrow \mathscr{W}^k(\mathfrak{g}, f_1)$. The bilinear form $\tau_k^{(1)}$ is identically zero on the nilpotent subalgebra $\mathfrak{n}_0$ of $\mathfrak{g}^{\natural,1}$, the lemma follows. $\square$

Because the restriction $\overline{\chi}_0$ of the linear form χ_2 to the Lie subalgebra $\mathfrak{n}_2$ is a character, we get a vertex algebra map

$$\Upsilon_0 : \mathscr{V}(\mathfrak{n}_0) \longrightarrow \mathscr{W}^k(\mathfrak{g}, f_1)$$

defined for a free generator x in $\mathfrak{n}_0$ by

$$\Upsilon_0(x) \coloneqq \begin{cases} J^{\{x\}} + \chi_2(x)\mathbf{1} & \text{if} \quad x \in \mathfrak{g}_{0,2} \\ J^{\{x\}} & \text{otherwise.} \end{cases}$$

We state Theorem 4.

Theorem 6.2.1.3. *Assume that Conditions ($\bigstar$) hold. Following Section 5.3.3, to the chiral comoment map $\Upsilon_0 : \mathscr{V}(\mathfrak{n}_0) \to \mathscr{W}^k(\mathfrak{g}, f_1)$ corresponds a BRST cochain complex $(\mathscr{C}_0^\bullet, d_0)$, where $\mathscr{C}_0 \coloneqq \mathscr{W}^k(\mathfrak{g}, f_1) \otimes_{\mathbf{C}} \mathscr{F}^\bullet(\mathfrak{n}_0 \oplus \mathfrak{n}_0{}^*)$.*

Then the cohomology of this complex is concentrated in degree 0 and isomorphic to the affine W-algebra $\mathscr{W}^k(\mathfrak{g}, f_2)$:

$$\mathrm{H}^\bullet(\mathscr{C}_0^\bullet, d_0) \cong \delta_{\bullet=0}\, \mathscr{W}^k(\mathfrak{g}, f_2).$$

To prove Theorem 6.2.1.3, we need to find a natural map

$$\mathrm{H}^0(\mathcal{C}_0^\bullet, d_0) \longrightarrow \mathcal{W}^k(\mathfrak{g}, f_2).$$

To do so, we need to introduce a new construction of $\mathcal{W}^k(\mathfrak{g}, f_2)$ that relies on the nilpotent algebra

$$\widetilde{\mathfrak{n}}_2 \coloneqq \mathfrak{g}^{(1)}_{\geqslant 1} \oplus \mathfrak{n}_0,$$

that satisfies the semi-direct product decomposition:

$$\widetilde{\mathfrak{n}}_2 = \widetilde{\mathfrak{n}}_1 \oplus \mathfrak{n}_0 \quad \text{and} \quad [\widetilde{\mathfrak{n}}_1, \mathfrak{n}_0] \subseteq \widetilde{\mathfrak{n}}_1.$$

Remark 6.2.1.4. In general, the Lie algebra $\widetilde{\mathfrak{n}}_2$ is not equal to $\mathfrak{g}^{(2)}_{\geqslant 1}$. In fact, it does not come from a good grading for f_2.

Example 6.2.1.5. Take $\mathfrak{g} = \mathfrak{sl}_4$ the Lie algebra of traceless square matrices of size 4. Denote by $E_{i,j}$ the elementary matrices. Take

$$f_1 \coloneqq \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & 0 & \\ 1 & & & 0 \end{pmatrix} \quad \text{and} \quad f_2 \coloneqq \begin{pmatrix} 0 & & & \\ & 0 & & \\ & 1 & 0 & \\ 1 & & & 0 \end{pmatrix}$$

the two nilpotent elements introduced in Example 2.3.1.5. One has:

$$\mathfrak{n}_1 = \left\{ \begin{pmatrix} 1 & 0 & * & * \\ & 1 & 0 & * \\ & & 1 & 0 \\ & & & 1 \end{pmatrix} \right\}, \quad \widetilde{\mathfrak{n}}_1 = \mathfrak{g}^{(1)}_{\geqslant 1} = \left\{ \begin{pmatrix} 1 & * & * & * \\ & 1 & 0 & * \\ & & 1 & * \\ & & & 1 \end{pmatrix} \right\},$$

$$\mathfrak{g}^{(2)}_{\geqslant 1} = \mathfrak{n}_2 = \left\{ \begin{pmatrix} 1 & 0 & * & * \\ & 1 & * & * \\ & & 1 & 0 \\ & & & 1 \end{pmatrix} \right\}, \quad \widetilde{\mathfrak{n}}_2 = \left\{ \begin{pmatrix} 1 & * & * & * \\ & 1 & * & * \\ & & 1 & * \\ & & & 1 \end{pmatrix} \right\}.$$

Pick a basis $\{x_i\}_{i \in I(\widetilde{\mathfrak{n}}_2)}$ of $\widetilde{\mathfrak{n}}_2$ which is the union of a basis $\{x_i\}_{i \in I(\widetilde{\mathfrak{n}}_1)}$ of $\widetilde{\mathfrak{n}}_1$ and a basis $\{x_i\}_{i \in I(\mathfrak{n}_0)}$ of $\mathfrak{n}_0$. Introduce the chiral comoment map

$$\widetilde{\Upsilon}_2 : \mathcal{V}(\widetilde{\mathfrak{n}}_2) \longrightarrow \mathcal{V}^k(\mathfrak{g}) \otimes_{\mathbf{C}} \mathcal{A}(\mathfrak{g}_1^{(1)})$$

defined on strong generators $x \in \widetilde{\mathfrak{n}}_2$ by:

$$\widetilde{\Upsilon}_2(x) \coloneqq \begin{cases} x + \dfrac{1}{2} \displaystyle\sum_{i=1}^{2s} :\psi^i \psi_{[v_i, x]}: + \chi_2(x)\mathbf{1} & \text{if} \quad x \in \mathfrak{n}_0 \\ x + \psi_x \quad \text{if} \quad x \in \mathfrak{g}_1^{(1)} \\ x + \chi_2(x)\mathbf{1} \quad \text{if} \quad x \in \mathfrak{g}_2^{(1)} \\ x \quad \text{otherwise.} \end{cases}$$

Denote by $\left(\widetilde{\mathcal{C}}_2^\bullet, \widetilde{d}_2\right)$ the associated BRST complex, where

$$\widetilde{\mathcal{C}}_2^\bullet \coloneqq \mathcal{V}^k(\mathfrak{g}) \otimes_{\mathbf{C}} \mathcal{A}(\mathfrak{g}_1^{(1)}) \otimes_{\mathbf{C}} \mathcal{F}^\bullet(\widetilde{\mathfrak{n}}_2 \oplus \widetilde{\mathfrak{n}}_2{}^*).$$

Recall that the differential $\widetilde{d}_2$ is the 0-th mode of the element

$$\widetilde{Q}_2 \coloneqq \sum_{i \in I(\widetilde{\mathfrak{n}}_2)} {:}\widetilde{\Upsilon}_2(x_i)\phi_i^*{:} - \frac{1}{2} \sum_{i,j \in I(\widetilde{\mathfrak{n}}_2)} {:}\phi_{[x_i,x_j]}\phi_i^*\phi_j^*{:}.$$

As a charge-graded vector space, the cochain complex has the tensor product decomposition

$$\widetilde{\mathcal{C}}_2^\bullet = \widetilde{\mathcal{C}}_1^\bullet \otimes_{\mathbf{C}} \mathcal{F}^\bullet(\mathfrak{n}_0 \oplus \mathfrak{n}_0{}^*).$$

Because of the embedding (6.2.1.1), one has a vertex superalgebra embedding

$$\mathcal{C}_0^\bullet \subseteq \widetilde{\mathcal{C}}_2^\bullet.$$

Recall that the differential d_0 on the cochain complex $\mathcal{C}_0^\bullet$ is given by the 0-th mode of the element

$$Q_0 \coloneqq \sum_{i \in I(\mathfrak{n}_0)} {:}\Upsilon_0(x_i)\phi_i^*{:} - \frac{1}{2} \sum_{i,j \in I(\mathfrak{n}_0)} {:}\phi_{[x_i,x_j]}\phi_i^*\phi_j^*{:}.$$

Lemma 6.2.1.6. *1. The decomposition $\widetilde{Q}_2 = \widetilde{Q}_1 + Q_0$ holds in $\widetilde{\mathcal{C}}_2^\bullet$. Hence, the decomposition $\widetilde{d}_2 = \widetilde{d}_1 + d_0$ holds as operators on $\widetilde{\mathcal{C}}_2^\bullet$.*

2. The λ-superbracket $\left[\widetilde{Q}_{1\lambda} Q_0\right]$ is zero. Hence, the superbracket $[\widetilde{d}_1, d_0]$ of operators is zero.

Proof. By definition, ${:}\phi_{[x_i,x_j]}\phi_i^*\phi_j^*{:} = {:}\phi_{[x_i,x_j]}({:}\phi_i^*\phi_j^*{:}){:}$. Because their λ-bracket is zero, one can permute the last two terms:

$${:}\phi_{[x_i,x_j]}\phi_i^*\phi_j^*{:} = -{:}\phi_{[x_i,x_j]}\phi_j^*\phi_i^*{:}.$$

Then it is clear that

$$\widetilde{Q}_2 = \widetilde{Q}_1 + \sum_{i \in I(\mathfrak{n}_0)} {:}\Big(x + \frac{1}{2}\sum_{i=1}^{2s} {:}\psi^i\psi_{[v_i,x]}{:} + \chi_2(x)\Big)\phi_i{:} + \sum_{\substack{i \in I(\mathfrak{n}_0) \\ j \in I(\widetilde{\mathfrak{n}}_1)}} {:}\phi_{[x_i,x_j]}\phi_j^*\phi_i^*{:}$$
$$- \frac{1}{2} \sum_{i,j \in I(\mathfrak{n}_0)} {:}\phi_{[x_i,x_j]}\phi_i^*\phi_j^*{:}.$$

We recall the associator formula [DSK06, (1.40)]:

$$\begin{aligned} &{:}({:}\phi_{[x_i,x_j]}\phi_j^*{:})\phi_i^*{:} - {:}\phi_{[x_i,x_j]}({:}\phi_j^*\phi_i^*{:}){:} \\ &\quad = \sum_{n \geqslant 0} \frac{1}{(n+1)!}\big({:}(\partial^{n+1}\phi_{[x_i,x_j]})(\phi_{j\,(n)}^*\phi_i^*){:} - {:}(\partial^{n+1}\phi_j^*)(\phi_{[x_i,x_j]\,(n)}\phi_i^*){:}\big). \end{aligned}$$

If $i \in I(\mathfrak{n}_0)$ and $j \in I(\widetilde{\mathfrak{n}}_1)$, then one has for all nonnegative integer n:

$$\phi^*_{j\,(n)}\phi^*_i = 0 \quad \text{and} \quad \phi_{[x_i,x_j]\,(n)}\phi^*_i = 0,$$

the second equality is due to the fact that $[x_i, x_j]$ belongs to $\widetilde{\mathfrak{n}}_1$. Hence if i is in $I(\mathfrak{n}_0)$ and j in $I(\widetilde{\mathfrak{n}}_1)$, then

$$:(:\phi_{[x_i,x_j]}\phi^*_j:)\phi^*_i: - :\phi_{[x_i,x_j]}(:\phi^*_j\phi^*_i:): = 0.$$

We deduce that

$$\widetilde{Q}_2 = \widetilde{Q}_1 + \sum_{i \in I(\mathfrak{n}_0)} :(J^{\{x_i\}} + \chi_2(x))\phi_i: - \frac{1}{2}\sum_{i,j \in I(\mathfrak{n}_0)} :\phi_{[x_i,x_j]}\phi^*_i\phi^*_j: = \widetilde{Q}_1 + Q_0.$$

According to [KRW03, Theorem 2.4(a)], one has $\left[\widetilde{Q}_{1\lambda} J^{\{x\}}\right] = 0$ for all x in $\mathfrak{n}_0$. It follows that $\left[\widetilde{Q}_{1\lambda} Q_0\right] = 0$. □

As a consequence of this lemma, the embedding (6.2.1.1) induces an embedding

$$\mathrm{Ker}(d_0) \cap C_0^n \subseteq \mathrm{Ker}(\widetilde{d}_2) \cap \widetilde{C}_2^n \quad \text{for} \quad n \in \mathbf{Z}.$$

Taking the cohomology, we get a natural map

$$\Theta : \mathrm{H}^\bullet(C_0^\bullet, d_0) \longrightarrow \mathrm{H}^\bullet\big(\widetilde{C}_2^\bullet, \widetilde{d}_2\big). \tag{6.2.1.7}$$

To prove Theorem 6.2.1.3, it is enough to show that this map is an isomorphism and to prove that the codomain is isomorphic to $\mathscr{W}^k(\mathfrak{g}, f_2)$.

Remark 6.2.1.8. If $\mathfrak{g}_1^{(1)} = \{0\}$, then $\mathfrak{n}_1 = \widetilde{\mathfrak{n}}_1$ and $\mathfrak{n}_2 = \widetilde{\mathfrak{n}}_2$. The construction is simplified because Theorem 5 is obvious in this this case: one can use the ideas of [MR97] and only needs homological algebra tools. To generalise the construction to the case $\mathfrak{g}_1^{(1)} \neq \{0\}$, we need to use some geometry.

6.2.2. New construction of the second Slodowy slice

In this section, we prove that the (arc space of) the Slodowy slice S_2 can be computed by a BRST cohomology associated to the group $\widetilde{\mathfrak{n}}_2$ (Corollary 6.2.2.8).

Lemma 6.2.2.1. *The inclusion* $[\widetilde{\mathfrak{n}}_2, \widetilde{\mathfrak{n}}_2] \subseteq \mathfrak{n}_2$ *holds.*

Proof. We use the decomposition $\widetilde{\mathfrak{n}}_2 = \widetilde{\mathfrak{n}}_1 \oplus \mathfrak{n}_0$. We have the inclusions

$$[\widetilde{\mathfrak{n}}_1, \widetilde{\mathfrak{n}}_1] \subseteq \mathfrak{g}^{(1)}_{\geqslant 2} \subseteq \mathfrak{n}_2 \quad \text{and} \quad [\mathfrak{n}_0, \mathfrak{n}_0] \subseteq \mathfrak{n}_0 \subseteq \mathfrak{n}_2.$$

It remains to show the inclusion $[\widetilde{\mathfrak{n}}_1, \mathfrak{n}_0] \subseteq \mathfrak{n}_2$.

Recall the decompositions $\widetilde{\mathfrak{n}}_1 = \mathfrak{g}_{1,0} \oplus (\mathfrak{g}^{(1)}_{\geqslant 1} \cap \mathfrak{g}^{(2)}_{\geqslant 1})$ and $\mathfrak{n}_0 = \mathfrak{a} \oplus (\mathfrak{g}^{(1)}_0 \cap \mathfrak{g}^{(2)}_{\geqslant 2})$. The inclusion $[\widetilde{\mathfrak{n}}_1, \mathfrak{n}_0] \subseteq \mathfrak{n}_2$ follows from the following inclusions:

$$
\begin{aligned}
{[\widetilde{\mathfrak{n}}_1, \mathfrak{g}^{(1)}_0 \cap \mathfrak{g}^{(2)}_{\geqslant 2}]} &\subseteq [\mathfrak{g}^{(2)}_{\geqslant 0}, \mathfrak{g}^{(2)}_{\geqslant 2}] \subseteq [\mathfrak{g}_{1,0}, \mathfrak{g}_{0,1}] \subseteq \mathfrak{n}_2,\\
[\mathfrak{g}^{(1)}_{\geqslant 1} \cap \mathfrak{g}^{(2)}_{\geqslant 1}, \mathfrak{a}] &\subseteq [\mathfrak{g}^{(1)}_{\geqslant 1} \cap \mathfrak{g}^{(2)}_{\geqslant 1}, \mathfrak{g}_{0,1}] \subseteq \mathfrak{g}^{(2)}_{\geqslant 2} \subseteq \mathfrak{n}_2,\\
[\mathfrak{g}_{1,0}, \mathfrak{a}] &\subseteq [\mathfrak{g}_{1,0}, \mathfrak{g}_{0,1}] \subseteq \mathfrak{g}_{1,1} \subseteq \mathfrak{n}_2.
\end{aligned}
$$

□

Take $i = 1, 2$. The nilpotent Lie algebra $\widetilde{\mathfrak{n}}_i$ is the Lie algebra of a unipotent subgroup $\widetilde{N}_i$ of G, which acts on $\mathfrak{g}^*$ by the coadjoint action, and this action is Hamiltonian with the moment map given by the restriction map

$$\widetilde{\pi}_i : \mathfrak{g}^* \longrightarrow \widetilde{\mathfrak{n}}_i{}^*, \quad \xi \longmapsto \xi|_{\widetilde{\mathfrak{n}}_i}.$$

Denote by $\widetilde{\mathbb{O}}_i \coloneqq \mathrm{Ad}^*(\widetilde{N}_i)\widetilde{\chi}_i$ the coadjoint orbit of the linear form $\widetilde{\chi}_i \coloneqq \chi_i|_{\widetilde{\mathfrak{n}}_i}$.

By the Gan–Ginzburg construction applied to $\widetilde{\mathfrak{n}}_1 = \mathfrak{g}^{(1)}_{\geqslant 1}$ (Subsection 2.2.3), there is a symplectic isomorphism

$$\widetilde{\sigma}_1 : \mathfrak{g}^{(1)}_1 \xrightarrow{\sim} \widetilde{\mathbb{O}}_1, \quad v \longmapsto \widetilde{\chi}_1 + \mathrm{ad}^*(v)\widetilde{\chi}_1.$$

Hence, the fibre of the orbit by the moment map is

$$\widetilde{\pi}_1{}^{-1}(\widetilde{\mathbb{O}}^-_1) = \chi_1 + \mathrm{ad}^*(\mathfrak{g}^{(1)}_1)\chi_1 \oplus (\mathfrak{g}^{(1)}_{\geqslant 1})^\perp.$$

There is a natural map given by the inclusion $\pi_1{}^{-1}(\mathbb{O}^-_1) \subseteq \widetilde{\pi}_1{}^{-1}\big(\widetilde{\mathbb{O}}^-_1\big)$. This inclusion is N_0-equivariant, each subspace is indeed N_0-stable because of the semidirect product decompositions. By Corollary 2.2.5.1, this inclusion induces an isomorphism

$$\pi_1{}^{-1}(\mathbb{O}^-_1)/\!/N_1 \cong \widetilde{\pi}_1{}^{-1}\big(\widetilde{\mathbb{O}}^-_1\big)/\!/\widetilde{N}_1.$$

Moreover, the N_0-actions descend to both quotients and the isomorphism is N_0-equivariant.

The Lie algebra $\widetilde{\mathfrak{n}}_2$ is not covered by the Gan–Ginzburg construction in general. But analogue properties hold.

Proposition 6.2.2.2. *There is a symplectic isomorphism given by the map*

$$\widetilde{\sigma}_2 : \mathfrak{g}^{(1)}_1 \xrightarrow{\sim} \widetilde{\mathbb{O}}_2, \quad v \longmapsto \widetilde{\chi}_2 + \mathrm{ad}^*(v)\widetilde{\chi}_2 + \frac{1}{2}\,\mathrm{ad}^*(v)^2\widetilde{\chi}_2.$$

Moreover, the restriction map $\widetilde{\mathfrak{n}}_2{}^* \twoheadrightarrow \widetilde{\mathfrak{n}}_1{}^*$ induces a symplectic isomorphism between the coadjoint orbits $\widetilde{\mathbb{O}}_2$ and $\widetilde{\mathbb{O}}_1$. Whence, the following diagramm commutes:

$$
\begin{tikzcd}
 & \widetilde{\mathbb{O}}_2 \arrow[r, hook] \arrow[d, "\wr"'] & \widetilde{\mathfrak{n}}_2{}^* \arrow[d, two heads] \\
\mathfrak{g}^{(1)}_1 \arrow[ur, "\widetilde{\sigma}_2", "\sim"'] \arrow[dr, "\sim", "\widetilde{\sigma}_1"'] & & \\
 & \widetilde{\mathbb{O}}_1 \arrow[r, hook] & \widetilde{\mathfrak{n}}_1{}^*.
\end{tikzcd}
$$

Proof. Notice the equality $\mathrm{Ad}^*(N_0)\widetilde{\chi}_2 = \{\widetilde{\chi}_2\}$ which follows from the decomposition $f_2 = f_1 + f_0$, the facts that f_1 is a fixed point for the action of N_0 and that the restriction of $\widetilde{\chi}_2$ to $\mathfrak{n}_0$ is the character $\overline{\chi}_0$. Because there is the semidirect product decomposition $\widetilde{N}_2 = \widetilde{N}_1 \rtimes N_0$, one has

$$\widetilde{\mathcal{O}}_2 = \mathrm{Ad}^*(\widetilde{N}_1)\,\mathrm{Ad}^*(N_0)\widetilde{\chi}_2 = \mathrm{Ad}^*(\widetilde{N}_1)\widetilde{\chi}_2.$$

The action of group $\widetilde{N}_1$ is the exponential of the action of its Lie algebra $\widetilde{\mathfrak{n}}_1$:

$$\mathrm{Ad}^*(\widetilde{N}_1)\widetilde{\chi}_2 = \Big\{\sum_{k\geqslant 0} \frac{1}{k!}\,\mathrm{ad}^*(v)^k\widetilde{\chi}_2 \mid v \in \widetilde{\mathfrak{n}}_1\Big\}.$$

Take $k \geqslant 3$ and v in $\widetilde{\mathfrak{n}}_1$. Since f_2 is in $\mathfrak{g}^{(1)}_{\geqslant -2}$, the element $\mathrm{ad}(v)^k(f_2)$ belongs to $\mathfrak{g}^{(1)}_{\geqslant k-2} \subseteq \mathfrak{g}^{(1)}_{\geqslant 1}$. The inclusion $\mathfrak{g}^{(1)}_{\geqslant 1} \subseteq \widetilde{\mathfrak{n}}_2{}^{\perp}$ implies that $(\mathrm{ad}(v)^k(f_2)|\widetilde{\mathfrak{n}}_2) = 0$. Hence,

$$\mathrm{ad}^*(v)^k\widetilde{\chi}_2 = 0.$$

Let us check that if v belongs to $\mathfrak{g}^{(1)}_{\geqslant 2}$ and $k \geqslant 1$, then $\mathrm{ad}^*(v)^k\widetilde{\chi}_2 = 0$. The element $\mathrm{ad}(v)^k f_0$ belongs to $\mathfrak{g}^{(1)}_{\geqslant 2k} \subseteq \mathfrak{g}^{(1)}_{\geqslant 2}$ that is orthogonal to $\widetilde{\mathfrak{n}}_2 \subseteq \mathfrak{g}^{(1)}_{\geqslant 0}$. The element $\mathrm{ad}(v)^k f_1$ belongs to $\mathfrak{g}^{(1)}_{\geqslant 2(k-1)}$, which is included in $\mathfrak{g}^{(1)}_{\geqslant 2}$ for $k \geqslant 2$.

It remains to look at $\mathrm{ad}(v)^k f_1$ when $k = 1$. We use the decomposition:

$$\widetilde{\mathfrak{n}}_2 = \mathfrak{g}^{(2)}_{\geqslant 2} \oplus \mathfrak{a} \oplus \mathfrak{g}_{1,1} \oplus \mathfrak{g}_{2,1} \oplus \mathfrak{g}_{1,0}.$$

Because f_1 is in $\mathfrak{g}_{-2,-2}$ and $\mathfrak{g}^{(1)}_{\geqslant 2} \subseteq \mathfrak{g}^{(2)}_{\geqslant 1}$, the element $\mathrm{ad}(v)f_1$ belongs to the intersection $\mathfrak{g}^{(1)}_{\geqslant 0} \cap \mathfrak{g}^{(2)}_{\geqslant -1}$. The only way $\mathrm{ad}(v)f_1$ may not be orthogonal to $\widetilde{\mathfrak{n}}_2$ is when v is in $\mathfrak{g}_{2,1}$. But, by construction of $\mathfrak{a}$, one has $\omega_2(v, \mathfrak{a} \oplus \mathfrak{g}_{2,1}) = 0$, so $\mathrm{ad}^*(v)f_1$ is orthogonal to $\mathfrak{a}$ and then to $\widetilde{\mathfrak{n}}_2$.

This proves that

$$\widetilde{\sigma}_2 : \mathfrak{g}^{(1)}_1 \longrightarrow \widetilde{\mathfrak{n}}_2{}^*, \quad v \longmapsto \widetilde{\chi}_2 + \mathrm{ad}^*(v)\widetilde{\chi}_2 + \frac{1}{2}\,\mathrm{ad}^*(v)^2\widetilde{\chi}_2.$$

is a well-defined surjection. It is clear that the composition $\pi \circ \widetilde{\sigma}_2 = \widetilde{\sigma}_1$ because the projection $\widetilde{\mathfrak{n}}_2{}^* \twoheadrightarrow \widetilde{\mathfrak{n}}_1{}^*$ is $\widetilde{N}_1$-equivariant. This proves that $\widetilde{\sigma}_2$ is an isomorphism and that the projection restricts to an isomorphism $\widetilde{\mathcal{O}}_2 \cong \widetilde{\mathcal{O}}_1$. $\square$

Corollary 6.2.2.3. *The orbit $\widetilde{\mathcal{O}}_2$ is described by the formula*

$$\widetilde{\mathcal{O}}_2 = \Big\{\sum_{k=0}^{2} \frac{1}{k!}\,\mathrm{ad}^*(v)^k\widetilde{\chi}_2 \mid v \in \mathfrak{g}_{1,0} \oplus \mathfrak{g}_{1,1}\Big\} \oplus \mathrm{ad}^*(\mathfrak{g}_{1,2})\widetilde{\chi}_2.$$

Hence, the fibre of the orbit $\widetilde{\mathcal{O}}_2$ by the moment map is

$$\widetilde{\pi}_2{}^{-1}(\widetilde{\mathcal{O}}_2^-) = \Big\{-\sum_{k=0}^{2} \frac{1}{k!}\,\mathrm{ad}^*(v)^k\chi_2 \mid v \in \mathfrak{g}_{1,0} \oplus \mathfrak{g}_{1,1}\Big\} \oplus \mathrm{ad}^*(\mathfrak{g}_{1,2})\chi_2 \oplus \widetilde{\mathfrak{n}}_2{}^{\perp}.$$

Proof. Take v in $\mathfrak{g}_{1,0} \oplus \mathfrak{g}_{1,1}$ and w in $\mathfrak{g}_{1,2}$. One has

$$\mathrm{ad}^*(v+w)^2\widetilde{\chi}_2 = \mathrm{ad}^*(v)^2\widetilde{\chi}_2 + \mathrm{ad}^*(w)^2\widetilde{\chi}_2 + \mathrm{ad}^*(v)\,\mathrm{ad}^*(w)\widetilde{\chi}_2 + \mathrm{ad}^*(w)\,\mathrm{ad}^*(v)\widetilde{\chi}_2.$$

It is clear that the elements $\mathrm{ad}(w)^2 f_2$, $\mathrm{ad}(v)\,\mathrm{ad}(w) f_2$ and $\mathrm{ad}(w)\,\mathrm{ad}(v) f_2$ belong to $\mathfrak{g}^{(1)}_{\geqslant 0} \cap \mathfrak{g}^{(2)}_{\geqslant 0}$ and the latter subspace is contained in the orthogonal subspace $\widetilde{\mathfrak{n}}_2{}^{\perp,(\bullet|\bullet)}$, hence we get the equality $\mathrm{ad}^*(v+w)^2\widetilde{\chi}_2 = \mathrm{ad}^*(v)^2\widetilde{\chi}_2$. Then, the corollary follows immediately from Proposition 6.2.2.2. □

Remark 6.2.2.4. Note that the description of the orbit $\widetilde{\mathbb{O}}_2$ needs quadric terms in the parameter v in $\mathfrak{g}_{1,0} \oplus \mathfrak{g}_{1,1}$, contrary to $\widetilde{\mathbb{O}}_1$.

We compute the corresponding Hamiltonian reduction by using the fact that the Hamiltonian action of N_2 is well-understood, and the reduction by stages as well.

Lemma 6.2.2.5. *The inclusion* $\pi_2{}^{-1}(\mathbb{O}_2^-) \subseteq \widetilde{\pi}_2^{-1}\big(\widetilde{\mathbb{O}}_2^-\big)$ *holds.*

Proof. By construction, there is the decomposition $\widetilde{\mathfrak{n}}_2 = \mathfrak{n}_2 \oplus \mathfrak{g}_{1,0}$ of vector spaces. Then, the orthogonal of $\mathfrak{n}_2$ for $(\bullet|\bullet)$ is $\mathfrak{n}_2{}^{\perp,(\bullet|\bullet)} = \widetilde{\mathfrak{n}}_2{}^{\perp,(\bullet|\bullet)} \oplus \mathfrak{g}_{-1,0}$. Using the fact that $\mathrm{ad}(f_1)$ induces an isomorphism $\mathfrak{g}_{1,2} \cong \mathfrak{g}_{-1,0}$, and the isomorphism $\mathfrak{g} \cong \mathfrak{g}^*$ induced by the bilinear form $(\bullet|\bullet)$, we deduce the equality

$$\widetilde{\mathfrak{n}}_2{}^{\perp} \oplus \mathrm{ad}^*(\mathfrak{g}_{1,2})\chi_1 = \mathfrak{n}_2{}^{\perp}.$$

We have the inclusion $[f_2, \mathfrak{g}_{1,2}] \subseteq \mathfrak{g}_{-1,0} \oplus \mathfrak{g}_{1,0}$ and the adjoint action of f_2 is injective on $\mathfrak{g}_{1,2}$. Because of the inclusion $\mathfrak{g}_{1,0} \subseteq \widetilde{\mathfrak{n}}_2$, we deduce that

$$\mathfrak{n}_2{}^{\perp} = \widetilde{\mathfrak{n}}_2{}^{\perp} \oplus \mathrm{ad}^*(\mathfrak{g}_{1,2})\chi_1 = \widetilde{\mathfrak{n}}_2{}^{\perp} \oplus \mathrm{ad}^*(\mathfrak{g}_{1,2})\chi_2.$$

One has the equality $\mathrm{ad}^*(\mathfrak{g}_{1,1})\chi_2 = \mathrm{ad}^*(\mathfrak{g}_{1,1})\chi_1$ since $[f_0, \mathfrak{g}_{1,1}] \subseteq \mathfrak{g}_{1,-1} = 0$. We get the explicit descriptions of both fibres:

$$\begin{aligned}
\pi_2{}^{-1}(\mathbb{O}_2^-) &= -\chi_2 + \mathrm{ad}^*(\mathfrak{g}_{1,1})\chi_2 \oplus \mathfrak{n}_2{}^{\perp},\\
\widetilde{\pi}_2{}^{-1}\big(\widetilde{\mathbb{O}}_2^-\big) &= \Big\{ -\chi_2 - \mathrm{ad}^*(v)\chi_2 - \mathrm{ad}^*(w)\chi_2\\
&\qquad - \frac{1}{2}\,\mathrm{ad}^*(v+w)^2\chi_2 \mid v \in \mathfrak{g}_{1,0},\ w \in \mathfrak{g}_{1,1}\Big\} \oplus \mathfrak{n}_2{}^{\perp}.
\end{aligned}$$

The inclusion is now clear. □

Theorem 6.2.2.6. *The inclusion of Lemma 6.2.2.5 induces a map*

$$\pi_2{}^{-1}(\mathbb{O}_2^-)/\!/N_2 \longrightarrow \widetilde{\pi}_2{}^{-1}\big(\widetilde{\mathbb{O}}_2^-\big)/\!/\widetilde{N}_2$$

which is a Poisson isomorphism.

For the proof, we need the following analogue of Lemma 2.3.3.5.

Lemma 6.2.2.7. *The isomorphism given by Theorem 2.2.3.3,*

$$\widetilde{\alpha}_1 : \widetilde{N}_1 \times S_1 \xrightarrow{\sim} \widetilde{\pi}_1^{-1}\big(\widetilde{\mathbb{O}}_1^-\big), \quad (g, \xi) \longmapsto \mathrm{Ad}^*(g)\xi,$$

restricts to an isomorphism

$$\widetilde{N}_1 \times (-\chi_2 + [\mathfrak{g}, e_1]^\perp \cap \mathfrak{n}_0{}^\perp) \cong \widetilde{\pi}_2{}^{-1}\big(\widetilde{\mathbb{O}}_2^-\big).$$

Proof. There is an inclusion $-\chi_2 + [\mathfrak{g}, e_1]^\perp \cap \mathfrak{n}_0{}^\perp \subseteq \widetilde{\pi}_2^{-1}\big(\widetilde{\mathbb{O}}_2^-\big)$ and the right-hand side variety is $\widetilde{N}_1$-stable, hence there is a closed embedding

$$\widetilde{\alpha}_1\big(\widetilde{N}_1 \times (-\chi_2 + [\mathfrak{g}, e_1]^\perp \cap \mathfrak{n}_0{}^\perp)\big) \subseteq \widetilde{\pi}_2^{-1}\big(\widetilde{\mathbb{O}}_2^-\big).$$

Both varieties are irreducible, so if their dimensions coincide then they are equal and the lemma is proved. The right-hand side has dimension

$$\begin{aligned}\dim(\widetilde{\pi}_2{}^{-1}\big(\widetilde{\mathbb{O}}_2^-\big)) &= \dim \mathfrak{g}_1^{(1)} + \dim(\widetilde{\mathfrak{n}}_2)^\perp = \dim \mathfrak{g}_1^{(1)} + \dim \mathfrak{g} - \dim \widetilde{\mathfrak{n}}_2 \\ &= \dim \mathfrak{g}_1^{(1)} + \dim \mathfrak{g} - \dim \widetilde{\mathfrak{n}}_1 - \dim \mathfrak{n}_0.\end{aligned}$$

Because $\widetilde{\alpha}_1$ is an isomorphism, we have $\dim \widetilde{\mathfrak{n}}_1 + \dim \mathfrak{g}^{e_1} = \dim \mathfrak{g} - \dim \widetilde{\mathfrak{n}}_1 + \dim \mathfrak{g}_1^{(1)}$. Hence

$$\dim(\widetilde{\pi}_2{}^{-1}\big(\widetilde{\mathbb{O}}_2^-\big)) = \dim \mathfrak{g}^{e_1} - \dim \mathfrak{n}_0 + \dim \widetilde{\mathfrak{n}}_1.$$

Moreover, we proved in Lemma 2.3.3.5 that

$$\dim \mathfrak{g}^{e_1} - \dim \mathfrak{n}_0 = \dim[\mathfrak{g}, e_1]^\perp \cap \mathfrak{n}_0{}^\perp,$$

whence

$$\dim \mathfrak{g}^{e_1} - \dim \mathfrak{n}_0 + \dim \widetilde{\mathfrak{n}}_1 = \dim \widetilde{\mathfrak{n}}_1 + \dim[\mathfrak{g}, e_1]^\perp \cap \mathfrak{n}_0{}^\perp.$$

This is the dimension of $\widetilde{\alpha}_1\big(\widetilde{N}_1 \times (-\chi_2 + [\mathfrak{g}, e_1]^\perp \cap \mathfrak{n}_0{}^\perp)\big)$, hence

$$\dim \widetilde{\alpha}_1\big(\widetilde{N}_1 \times (-\chi_2 + [\mathfrak{g}, e_1]^\perp \cap \mathfrak{n}_0{}^\perp)\big) = \dim(\widetilde{\pi}_2{}^{-1}\big(\widetilde{\mathbb{O}}_2\big)).$$

□

Proof of Theorem 6.2.2.6. Consider the affine GIT quotients:

$$\begin{aligned}\pi_2{}^{-1}(\mathbb{O}_2^-)/\!/N_2 &= \mathrm{Spec}\,\mathbf{C}[\pi_2{}^{-1}(\mathbb{O}_2^-)]^{N_2}, \\ \widetilde{\pi}_2^{-1}\big(\widetilde{\mathbb{O}}_2^-\big)/\!/\widetilde{N}_2 &= \mathrm{Spec}\,\mathbf{C}\big[\widetilde{\pi}_2^{-1}\big(\widetilde{\mathbb{O}}_2^-\big)\big]^{\widetilde{N}_2}.\end{aligned}$$

We have a $\mathbf{C}$-algebra map

$$\mathbf{C}\big[\widetilde{\pi}_2{}^{-1}\big(\widetilde{\mathbb{O}}_2^-\big)\big]^{\widetilde{N}_2} \longrightarrow \mathbf{C}[\pi_2{}^{-1}(\mathbb{O}_2^-)]^{N_2},$$

which clearly respect the Poisson structures on both sides.

The semi-direct product decompositions $N_2 = N_1 \rtimes N_0$ and $\widetilde{N}_2 = \widetilde{N}_1 \rtimes N_0$ imply:

$$\begin{aligned}
\mathbf{C}\big[\widetilde{\pi}_2^{-1}\big(\widetilde{\mathbb{O}}_2^-\big)\big]^{\widetilde{N}_2} &= \Big(\mathbf{C}\big[\widetilde{\pi}_2^{-1}\big(\widetilde{\mathbb{O}}_2^-\big)\big]^{\widetilde{N}_1}\Big)^{N_0},\\
\mathbf{C}[{\pi_2}^{-1}(\mathbb{O}_2^-)]^{N_2} &= \big(\mathbf{C}[{\pi_2}^{-1}(\mathbb{O}_2^-)]^{N_1}\big)^{N_0}.
\end{aligned}$$

We apply Lemmas 2.3.3.5 and 6.2.2.7, so the algebra map

$$\mathbf{C}\big[\widetilde{\pi}_2^{-1}\big(\widetilde{\mathbb{O}}_2^-\big)\big]^{\widetilde{N}_1} \longrightarrow \mathbf{C}[{\pi_2}^{-1}(\mathbb{O}_2^-)]^{N_1}.$$

is an N_0-isomorphism. After taking the N_0-invariants, we get the desired isomorphism. □

Corollary 6.2.2.8. *The Lie algebra cohomology with coefficients in the $\widetilde{\mathfrak{n}}_2[t]$-module $\mathbf{C}\big[\mathrm{J}_\infty{\widetilde{\pi}_2}^{-1}\big(\widetilde{\mathbb{O}}_2^-\big)\big]$ is given by the superalgebra isomorphism*

$$\mathrm{H}^\bullet\Big(\widetilde{\mathfrak{n}}_2[t], \mathbf{C}\big[\mathrm{J}_\infty{\widetilde{\pi}_2}^{-1}\big(\widetilde{\mathbb{O}}_2^-\big)\big]\Big) \cong \delta_{\bullet=0}\,\mathbf{C}[\mathrm{J}_\infty S_2],$$

where the isomorphism is induced by the action map

$$\widetilde{N}_2 \times S_2 \longrightarrow {\widetilde{\pi}_2}^{-1}\big(\widetilde{\mathbb{O}}_2^-\big), \quad (g,\xi) \longmapsto \mathrm{Ad}^*(g)\xi.$$

Proof. For m a nonnegative integer, consider the m-jets. Because of the decomposition $\widetilde{N}_2 = \widetilde{N}_1 \rtimes N_0$, we can compute $\mathrm{H}^\bullet\Big(\widetilde{\mathfrak{n}}_2[t]/(t^{m+1}), \mathbf{C}\big[\mathrm{J}_m{\widetilde{\pi}_2}^{-1}\big(\widetilde{\mathbb{O}}_2^-\big)\big]\Big)$ using the associated Hochschild–Serre spectral sequence $\{E_r\}_{r=0}^\infty$. The 1-st page is given by

$$E_1^{p,q} = \mathrm{H}^q\Big(\widetilde{\mathfrak{n}}_1[t]/(t^{m+1}), \mathbf{C}\big[\mathrm{J}_m{\widetilde{\pi}_2}^{-1}\big(\widetilde{\mathbb{O}}_2^-\big)\big]\Big) \otimes_{\mathbf{C}} \bigwedge^p \Big(\big(\mathfrak{n}_0[t]/(t^{m+1})\big)^*\Big).$$

The 2-nd page is given by

$$E_2^{p,q} = \mathrm{H}^p\bigg(\mathfrak{n}_0[t]/(t^{m+1}), \mathrm{H}^q\Big(\widetilde{\mathfrak{n}}_1[t]/(t^{m+1}), \mathbf{C}\big[\mathrm{J}_m{\widetilde{\pi}_2}^{-1}\big(\widetilde{\mathbb{O}}_2^-\big)\big]\Big)\bigg).$$

The infinity page is

$$E_\infty^{p,q} = \mathrm{gr}^p\,\mathrm{H}^{p+q}\Big(\widetilde{\mathfrak{n}}_2[t]/(t^{m+1}), \mathbf{C}\big[\mathrm{J}_m{\widetilde{\pi}_2}^{-1}\big(\widetilde{\mathbb{O}}_2^-\big)\big]\Big),$$

where the implicit filtration is the one induced by the double complex structure. The double complex is bounded so the spectral sequence is convergent.

Lemma 6.2.2.7 implies the following isomorphism of $\widetilde{\mathfrak{n}}_1[t]/(t^{m+1})$-modules:

$$\mathbf{C}\big[\mathrm{J}_m{\widetilde{\pi}_2}^{-1}\big(\widetilde{\mathbb{O}}_2^-\big)\big] \cong \mathbf{C}\big[\mathrm{J}_m\widetilde{N}_1\big] \otimes_{\mathbf{C}} \mathbf{C}[\mathrm{J}_m(-\chi_2 + [\mathfrak{g}, e_1]^\perp \cap {\mathfrak{n}_0}^\perp)].$$

The Lie algebra cohomology of $\mathbf{C}\big[\mathrm{J}_m\widetilde{N}_1\big]$ is the algebraic de Rham cohomology of $\mathrm{J}_m\widetilde{N}_1$, and because this groups is unipotent, its cohomology is $\mathbf{C}$ in degree 0 and zero otherwise.

Consider the moment map $\mu_0 : \widetilde{\pi}_1{}^{-1}(\widetilde{\mathbb{O}}_1)/\!/\widetilde{N}_1 \to \mathfrak{n}_0{}^*$ defined by applying the reduction by stages construction (Lemma 2.1.4.3) to $\widetilde{\pi}_2$. Lemma 6.2.2.7 implies the N_0-equivariant isomorphism $\widetilde{\pi}_2{}^{-1}(\widetilde{\mathbb{O}}_2^-)/\!/\widetilde{N}_1 \cong \mu_0{}^{-1}(0)$, hence the first page component $E_1^{p,q}$ is zero when $q \neq 0$ and

$$E_1^{p,0} \cong \mathbf{C}\big[\mathrm{J}_m\mu_0{}^{-1}(0)\big] \otimes_{\mathbf{C}} \textstyle\bigwedge^p \Big(\big(\mathfrak{n}_0[t]/(t^{m+1})\big)^*\Big).$$

Thanks to Theorem 2.3.3.3, we can use the N_0-isomorphism $N_0 \times S_2 \cong \mu_0{}^{-1}(0)$ and the same argument as previously to say that $E_2^{p,q}$ is zero unless $p = q = 0$, so the spectral sequence collapses.

Then we deduce the isomorphism

$$\mathrm{H}^0\Big(\widetilde{\mathfrak{n}}_2[t]/(t^{m+1}), \mathbf{C}\big[\mathrm{J}_m\widetilde{\pi}_2{}^{-1}(\widetilde{\mathbb{O}}_2^-)\big]\Big) \cong \mathbf{C}[\mathrm{J}_m S_2],$$

and the other cohomology groups are trivial.

Consider the Lie algebra cochain complex

$$C_m \coloneqq \mathbf{C}\big[\mathrm{J}_m\widetilde{\pi}_2{}^{-1}(\widetilde{\mathbb{O}}_2^-)\big] \otimes_{\mathbf{C}} \textstyle\bigwedge^\bullet \Big(\big(\widetilde{\mathfrak{n}}_2[t]/(t^{m+1})\big)^*\Big)$$

for all $m \geqslant 0$. The sequence $\{C_m\}_{m\geqslant 0}$ form an inductive sequence and its colimit is the Lie algebra cochain complex of the $\widetilde{\mathfrak{n}}_2[t]$-module $\mathbf{C}\big[\mathrm{J}_\infty\widetilde{\pi}_2{}^{-1}(\widetilde{\mathbb{O}}_2^-)\big]$:

$$C_\infty \coloneqq \mathbf{C}\big[\mathrm{J}_\infty\widetilde{\pi}_2{}^{-1}(\widetilde{\mathbb{O}}_2^-)\big] \otimes_{\mathbf{C}} \textstyle\bigwedge_\infty^\bullet (\widetilde{\mathfrak{n}}_2{}^*).$$

Because taking the cohomology commutes with inductive colimits, one gets

$$\mathrm{H}^0\Big(\widetilde{\mathfrak{n}}_2[t], \mathbf{C}\big[\mathrm{J}_\infty\widetilde{\pi}_2{}^{-1}(\widetilde{\mathbb{O}}_2^-)\big]\Big) \cong \mathrm{colim}_{m\geqslant 0}\, \mathbf{C}[\mathrm{J}_m S_2] \cong \mathbf{C}[\mathrm{J}_\infty S_2].$$

□

Remark 6.2.2.9. We conjecture that the action map

$$\widetilde{N}_2 \times S_2 \longrightarrow \widetilde{\pi}_2{}^{-1}(\widetilde{\mathbb{O}}_2^-), \quad (g, \xi) \longmapsto \mathrm{Ad}^*(g)\xi$$

is an isomorphism. Such isomorphism should be used to apply Theorem 6. In fact, Corollary 6.2.2.8 is enough.

6.2.3. New construction of the second affine W-algebra

In this section, we prove that the W-algebra $\mathscr{W}^k(\mathfrak{g}, f_2)$ can be computed by a BRST cohomology associated to the group $\widetilde{\mathfrak{n}}_2$ (Theorem 6.2.3.7).

The Lie algebra $\mathfrak{n}_2$ corresponds to the choice of the isotropic subspace of $\mathfrak{l}_2$ in the construction of the affine W-algebra $\mathscr{W}^k(\mathfrak{g}, f_2)$ described in Subsection 6.1.1. Because of the isomorphism $\mathfrak{g}_{1,1} \cong \mathfrak{l}_2{}^{\perp,\omega_2}/\mathfrak{l}_2$, we have

$$\mathscr{W}^k(\mathfrak{g}, f_2) = \mathrm{H}^0(C_2^\bullet, d_2), \quad \text{where} \quad C_2^\bullet \coloneqq \mathscr{V}^k(\mathfrak{g}) \otimes_{\mathbf{C}} \mathscr{A}(\mathfrak{g}_{1,1}) \otimes_{\mathbf{C}} \mathscr{F}^\bullet(\mathfrak{n}_2 \oplus \mathfrak{n}_2{}^*).$$

Geometrically, the complex $(C_2^\bullet, d_2)$ corresponds to the moment map

$$\mu_2 : \mathfrak{g}^* \times \mathfrak{g}_{1,1} \longrightarrow {\mathfrak{n}_2}^*, \quad (\xi, v) \longmapsto \pi_2(\xi) + \overline{\chi}_2 + \mathrm{ad}^*(v)\overline{\chi}_2,$$

where the acting group is N_2. One has the N_2-equivariant isomorphism

$${\mu_2}^{-1}(0) \cong {\pi_2}^{-1}(\mathbb{O}_2^-).$$

The complex $\big(\widetilde{C}_2^\bullet, \widetilde{d}_2\big)$ corresponds to the moment map

$$\widetilde{\mu}_2 : \mathfrak{g}^* \times \mathfrak{g}_1^{(1)} \longrightarrow {\widetilde{\mathfrak{n}}_2}^*, \quad (\xi, v) \longmapsto \pi_2(\xi) + \widetilde{\chi}_2 + \mathrm{ad}^*(v)\widetilde{\chi}_2 + \frac{1}{2}\,\mathrm{ad}^*(v)\widetilde{\chi}_2,$$

where the acting group is $\widetilde{N}_2$. One has the $\widetilde{N}_2$-equivariant isomorphism

$${\widetilde{\mu}_2}^{-1}(0) \cong {\widetilde{\pi}_2}^{-1}\big(\widetilde{\mathbb{O}}_2^-\big).$$

Denote by $\big(\widetilde{C}_{2,\infty}^\bullet, \widetilde{d}_{2,\infty}\big)$ the Poisson vertex BRST complex associated with the arc space of this moment map.

Proposition 6.2.3.1. *The following cohomology vanishes outside degree* 0*:*

$$\mathrm{H}^n\big(\widetilde{C}_2^\bullet, \widetilde{d}_2\big) = 0 \quad \textit{for} \quad n \neq 0,$$

and there is a natural isomorphism

$$\mathrm{gr}^{\mathrm{F}}\, \mathrm{H}^0\big(\widetilde{C}_2^\bullet, \widetilde{d}_2\big) \xrightarrow{\sim} \mathrm{H}^0\big(\widetilde{C}_{2,\infty}^\bullet, \widetilde{d}_{2,\infty}\big),$$

where the filtration F *on the cohomology* $\mathrm{H}^0\big(\widetilde{C}_2^\bullet, \widetilde{d}_2\big)$ *is induced by the Li filtration on the complex* $\big(\widetilde{C}_2^\bullet, \widetilde{d}_2\big)$.

Proof. Consider the Hamiltonian operator $\widetilde{\mathscr{H}}_2^{\mathrm{old}} := \mathscr{H}^{\mathfrak{g}} + L_0^{\mathscr{A}}$ on the vertex algebra $\mathcal{V}^k(\mathfrak{g}) \otimes_{\mathbf{C}} \mathscr{A}(\mathfrak{g}_1^{(1)})$. It induces a nonnegative grading. The homogeneous subspaces are all finite dimensional and for any $x \in \widetilde{\mathfrak{n}}_2$, the image $\widetilde{\Upsilon}_2(a)$ is the sum of terms of degrees less than one. The standard comoment map is given by:

$$\widetilde{\Upsilon}_{2,\mathrm{st}}(x) := \begin{cases} x + \dfrac{1}{2}\displaystyle\sum_{i=1}^{2s} {:}\psi^i \psi_{[v_i,x]}{:} & \text{if} \quad x \in \mathfrak{n}_0, \\ x \quad \text{if} \quad x \in \widetilde{\mathfrak{n}}_1. \end{cases}$$

The induced action of $\mathcal{U}(\widetilde{\mathfrak{n}}_2[t^{-1}]t^{-1})$ is free because $\mathcal{V}^k(\mathfrak{g}) \otimes_{\mathbf{C}} \mathscr{A}(\mathfrak{g}^{(1)})$ is freely generated by any basis of $\mathfrak{g}$ and any basis of $\mathfrak{g}^{(1)}$. Hence Theorem 5.3.4.2 applies and the cohomology of the complex $\big(\widetilde{C}_2^\bullet, \widetilde{d}_2\big)$ is isomorphic to the cohomology of the corresponding quotient complex $\big(\widetilde{C}_{2,+}^\bullet, \widetilde{d}_{2,+}\big)$.

Let $\mathfrak{l}$ be a Lagrangian subspace of $\mathfrak{g}_{1,1}$ and $\mathfrak{l}^c$ be a Lagrangian complement such that $\mathfrak{l}$ and $\mathfrak{l}^c$ are perfectly paired by the symplectic form. Then $\mathfrak{g}_{1,0} \oplus \mathfrak{l}$ and $\mathfrak{l}^c \oplus \mathfrak{g}_{1,2}$ are perfectly paired by the symplectic form ω_1.

Denote by $\{v_i\}_{i=1}^{2s}$ a basis of $\mathfrak{g}_1^{(1)}$ such that $v_i = v^{2s-i+1}$ for all $1 \leqslant i \leqslant s$,

$$\mathrm{Span}_{\mathbf{C}}\{v_i\}_{i=1}^{s_0} = \mathfrak{g}_{1,0}, \quad \mathrm{Span}_{\mathbf{C}}\{v_i\}_{i=s_0+1}^{s} = \mathfrak{l},$$
$$\mathrm{Span}_{\mathbf{C}}\{v_i\}_{i=2s-s_0}^{s+1} = \mathfrak{l}^{\mathrm{c}}, \quad \mathrm{Span}_{\mathbf{C}}\{v_i\}_{i=2s-s_0+1}^{2s} = \mathfrak{g}_{1,2}.$$

For $1 \leqslant i \leqslant s_0$, set $a_i \coloneqq -\frac{1}{2}$ and for $s_0+1 \leqslant i \leqslant s$, set $a_i \coloneqq 0$.

Choose a basis $\{x_i\}_{i \in I(\widetilde{\mathfrak{n}}_2)}$ of $\widetilde{\mathfrak{n}}_2$ which is homogeneous for the H_2-grading and denote by $\delta(x_i)$ the degree of x_i. Consider the Hamiltonian operator

$$\widetilde{\mathscr{H}}_2^{\mathrm{new}} = \mathscr{H}^{\mathfrak{g}} + \frac{1}{2}\partial H_{2(1)} + L^{\mathscr{A}}(a_\bullet)_0 + L^{\mathscr{F}}(m_\bullet)_0 \quad \text{where} \quad m_\bullet = \frac{1}{2}\delta(x_\bullet).$$

The degrees of the strong generators are

$$\begin{aligned}
&\Delta(x) = 1 - \frac{\delta}{2} \quad \text{for} \quad x \in \mathfrak{g}_\delta^{(2)} \quad \text{and} \quad \delta \in \mathbf{Z},\\
&\Delta(\psi_v) = 1 - \frac{\delta}{2} \quad \text{for} \quad v \in \mathfrak{g}_{1,\delta} \quad \text{and} \quad \delta \in \{0,1,2\},\\
&\Delta(\phi_i) = 1 - \frac{\delta(x_i)}{2} \quad \text{and} \quad \Delta(\phi_i^*) = \frac{\delta(x_i)}{2} \quad \text{for} \quad i \in I(\widetilde{\mathfrak{n}}_2).
\end{aligned}$$

The element $\widetilde{Q}_2$ is homogeneous of degree 1 for $\widetilde{\mathscr{H}}_2^{\mathrm{new}}$ and the quotient complex $\widetilde{\mathscr{C}}_{2,+}$ has an induced grading which is nonnegative.

The moment map $\widetilde{\mu}_2$ is smooth because its differential is surjective at any point. Therefore, we can apply Theorem 5.3.5.1 by replacing the assumption (5) by the vanishing result of Corollary 6.2.2.8, which is enough, and we are done. □

We want to build an intermediary complex to compare $(\mathscr{C}_2^\bullet, \mathscr{d}_2)$ and $\left(\widetilde{\mathscr{C}}_2^\bullet, \widetilde{\mathscr{d}}_2\right)$, by analogy with Theorem 6.1.2.6. To do so, we need the following lemma.

Lemma 6.2.3.2. *The image of $\mathcal{V}(\mathfrak{n}_2)$ by the chiral comoment map $\widetilde{\Upsilon}_2$ is contained in the vertex subalgebra $\mathcal{V}^k(\mathfrak{g}) \otimes_{\mathbf{C}} \mathscr{A}(\mathfrak{g}_{1,1} \oplus \mathfrak{g}_{1,2})$ of $\mathcal{V}^k(\mathfrak{g}) \otimes_{\mathbf{C}} \mathscr{A}(\mathfrak{g}_1^{(1)})$.*

Proof. We just need to check it for a strong generator x in $\mathfrak{n}_2 = \mathfrak{n}_1 \oplus \mathfrak{n}_0$. If x is in $\mathfrak{n}_1$, it is clear that $\widetilde{\Upsilon}_2(x)$ belongs to $\mathcal{V}^k(\mathfrak{g}) \otimes_{\mathbf{C}} \mathscr{A}(\mathfrak{l}_1^{\perp,\omega_1})$. For x in $\mathfrak{n}_0$, we need to check that the quadratic term $\sum_{i=1}^{2s} {:}\psi^i \psi_{[v_i,x]}{:}$ belongs to $\mathscr{A}(\mathfrak{g}_{1,1} \oplus \mathfrak{g}_{1,2})$.

According to the proof of Lemma 2.3.1.3, there is a basis $\{v_i\}_{i=1}^{2s}$ of $\mathfrak{g}_1^{(1)}$ such that

$$\mathrm{Span}_{\mathbf{C}}\{v_i\}_{i=1}^{s_0} = \mathfrak{g}_{1,0}, \quad \{v_i\}_{i=s_0+1}^{2s-s_0} = \mathfrak{g}_{1,1}, \qquad \mathrm{Span}_{\mathbf{C}}\{v_i\}_{i=2s-s_0+1}^{2s} = \mathfrak{g}_{1,2}$$

and $v_i = v^{2s-i+1}$ for all $1 \leqslant i \leqslant s_0$. In particular, $\mathfrak{l}_1^{\perp,\omega_1} = \mathfrak{g}_{1,1} \oplus \mathfrak{g}_{1,2}$ is spanned by $\{v^i\}_{i=1}^{2s-s_0}$.

The conditions (★) imply the inclusion $[\mathfrak{g}_{1,2}, \mathfrak{m}_0] \subseteq \mathfrak{g}_1^{(1)} \cap \mathfrak{g}_{\geqslant 3}^{(2)} = 0$, hence

$$\sum_{i=1}^{2s} {:}\psi^i \psi_{[v_i,x]}{:} = \sum_{i=1}^{2s-s_0} {:}\psi^i \psi_{[v_i,x]}{:} \in \mathscr{A}(\mathfrak{g}_{1,1} \oplus \mathfrak{g}_{1,2}).$$

□

Remark 6.2.3.3. The last lemma is coherent with the description of the orbit $\widetilde{\mathbb{O}}_2$ given in Proposition 6.2.2.3.

By restriction, we have a chiral comoment map

$$\Upsilon_{2,\mathrm{int}} : \mathcal{V}(\mathfrak{n}_2) \longrightarrow \mathcal{V}^k(\mathfrak{g}) \otimes_{\mathbf{C}} \mathcal{A}(\mathfrak{g}_{1,1} \oplus \mathfrak{g}_{1,2}).$$

As a consequence of the previous lemmas, we can define an intermediary BRST complex, denoted by $(\mathcal{C}^\bullet_{2,\mathrm{int}}, \mathscr{d}_{2,\mathrm{int}})$, where $\mathcal{C}_{2,\mathrm{int}}$ is the subalgebra of $\widetilde{\mathcal{C}}_2$ defined by

$$\mathcal{C}^\bullet_{2,\mathrm{int}} \coloneqq \mathcal{V}^k(\mathfrak{g}) \otimes_{\mathbf{C}} \mathcal{A}(\mathfrak{g}_{1,1} \oplus \mathfrak{g}_{1,2}) \otimes_{\mathbf{C}} \mathcal{F}^\bullet(\mathfrak{n}_2 \oplus \widetilde{\mathfrak{n}}_2{}^*),$$

and the operator $\mathscr{d}_{2,\mathrm{int}}$ is the restriction of the coboundary operator $\mathscr{d}_2$ to the subalgebra $\mathcal{C}_{2,\mathrm{int}}$.

By construction, one has there is a vertex algebra embedding map

$$\Pi_1 : (\mathcal{C}^\bullet_{2,\mathrm{int}}, \mathscr{d}_{2,\mathrm{int}}) \hookrightarrow \left(\widetilde{\mathcal{C}}^\bullet_2, \widetilde{\mathscr{d}}_2\right)$$

and it commutes with the differentials because of Lemma 6.2.3.2.

Lemma 6.2.3.4. *The projection maps* $\mathfrak{g}_{1,1} \oplus \mathfrak{g}_{1,2} \twoheadrightarrow \mathfrak{g}_{1,1}$ *and* $\widetilde{\mathfrak{n}}_2{}^* \twoheadrightarrow \mathfrak{n}_2{}^*$ *induce an surjective map*

$$\Pi_2 : (\mathcal{C}^\bullet_{2,\mathrm{int}}, \mathscr{d}_{2,\mathrm{int}}) \longrightarrow\!\!\!\!\!\rightarrow (\mathcal{C}^\bullet_2, \mathscr{d}_2)$$

of cochain complexes and vertex superalgebras.

Proof. We use the same notations as in the proof of Lemma 6.2.3.2. The projection is clearly a vertex algebra map. For x in $\mathfrak{n}_0$, the quadratic term

$$\sum_{i=1}^{2s-s_0} {:}\psi^i \psi_{[v_i,x]}{:}$$

in $\Upsilon(x)$ is killed by the projection because if $1 \leqslant i \leqslant s_0$, then ψ^i belongs to $\mathfrak{g}_{1,2}$ and if $s_0 + 1 \leqslant i \leqslant 2s - s_0$, then $v_i \in \mathfrak{g}_{1,1}$ and $[v_i, \mathfrak{n}_0] \subseteq \mathfrak{g}_1^{(1)} \cap \mathfrak{g}_{\geqslant 2}^{(1)} = \mathfrak{g}_{1,2}$. Hence the following diagram of vertex algebra homomorphisms commutes:

$$\begin{tikzcd}
 & \mathcal{V}^k(\mathfrak{g}) \otimes_{\mathbf{C}} \mathcal{A}(\mathfrak{g}_{1,1} \oplus \mathfrak{g}_{1,2}) \arrow[d, two heads] \\
\mathcal{V}(\mathfrak{n}_2) \arrow[ur, "\Upsilon_{2,\mathrm{int}}"] \arrow[dr, "\Upsilon_2"'] & \\
 & \mathcal{V}^k(\mathfrak{g}) \otimes_{\mathbf{C}} \mathcal{A}(\mathfrak{g}_{1,1}).
\end{tikzcd}$$

It implies that the projection Π_2 commutes with the differential. □

Let us denote by $\mathbb{O}_{2,\mathrm{int}} \coloneqq \mathrm{Ad}^* \left(\widetilde{N}_2\right)\overline{\chi}_2$ the $\widetilde{N}_2$-orbit in $\mathfrak{n}_2{}^*$ of the restriction of the linear form χ_2 to $\mathfrak{n}_2$.

Lemma 6.2.3.5. *The orbit $\mathbb{O}_{2,\mathrm{int}}$ is described by the formula*

$$\mathbb{O}_{2,\mathrm{int}} = \Big\{ \sum_{k=0}^{2} \frac{1}{k!} \operatorname{ad}^*(v)^k \widetilde{\chi}_2 \mid v \in \mathfrak{g}_{1,0} \oplus \mathfrak{g}_{1,1} \Big\}.$$

Proof. The projection map $\widetilde{\mathfrak{n}}_2{}^* \twoheadrightarrow \mathfrak{n}_2{}^*$ is $\widetilde{N}_2$-equivariant and the orbit $\mathbb{O}_{2,\mathrm{int}}$ is the image of $\widetilde{\mathbb{O}}_2$. Using the explicit description of the latter orbit in Corollary 6.2.2.3 and the inclusion $\operatorname{ad}^*(\mathfrak{g}_{1,2})\chi_2 \subseteq \mathfrak{n}_2{}^{\perp}$, the lemma follows. $\square$

The intermediary moment map is defined as

$$\begin{aligned} \mu_{2,\mathrm{int}} : \mathfrak{g}^* \times (\mathfrak{g}_{1,0} \oplus \mathfrak{g}_{1,1}) &\longrightarrow \mathfrak{n}_2{}^*, \\ (\xi, v) &\longmapsto \pi_2(\xi) + \overline{\chi}_2 + \operatorname{ad}^*(v)\overline{\chi}_2 + \frac{1}{2} \operatorname{ad}^*(v)^2 \overline{\chi}_2, \end{aligned}$$

it is a $\widetilde{N}_2$-equivariant homomorphism because N_2 is a normal subgroup of $\widetilde{N}_2$.

Lemma 6.2.3.6. *The projection map and the embedding*

$$\mathfrak{g}^* \times \mathfrak{g}_1^{(1)} \twoheadrightarrow \mathfrak{g}^* \times (\mathfrak{g}_{1,0} \oplus \mathfrak{g}_{1,1}) \quad \textit{and} \quad \mathfrak{g}^* \times (\mathfrak{g}_{1,1}) \hookrightarrow \mathfrak{g}^* \times (\mathfrak{g}_{1,0} \oplus \mathfrak{g}_{1,1})$$

induce Poisson isomorphisms between the corresponding Hamiltonian reductions:

$$\mu_2{}^{-1}(0)/\!/N_2 \cong \mu_{2,\mathrm{int}}{}^{-1}(0)/\!/\widetilde{N}_2 \cong \widetilde{\mu}_2{}^{-1}(0)/\!/\widetilde{N}_2.$$

Proof. The projection map $\mathfrak{g}^* \times \mathfrak{g}_1^{(1)} \twoheadrightarrow \mathfrak{g}^* \times (\mathfrak{g}_{1,0} \oplus \mathfrak{g}_{1,1})$ induces an $\widetilde{N}_2$-equivariant isomorphism $\widetilde{\mu}_2{}^{-1}(0) \cong \mu_{2,\mathrm{int}}{}^{-1}(0)$. It follows from the parametrization of the intermediary orbit $\mathbb{O}_{2,\mathrm{int}}$ given in Lemma 6.2.3.5. Then the lemma follows from Theorem 6.2.2.6. $\square$

Theorem 6.2.3.7. *These inclusion and projection maps of cochain complexes*

$$\Pi_1 : (C^\bullet_{2,\mathrm{int}}, d_{2,\mathrm{int}}) \hookrightarrow \big(\widetilde{C}^\bullet_2, \widetilde{d}_2\big) \quad \textit{and} \quad \Pi_2 : (C^\bullet_{2,\mathrm{int}}, d_{2,\mathrm{int}}) \twoheadrightarrow (C^\bullet_2, d_2)$$

induce vertex algebra isomorphisms between their cohomologies:

$$\mathrm{H}^0(C^\bullet_2, d_2) \cong \mathrm{H}^0(C^\bullet_{2,\mathrm{int}}, d_{2,\mathrm{int}}) \cong \mathrm{H}^0\big(\widetilde{C}^\bullet_2, \widetilde{d}_2\big),$$

and they are all isomorphic to $\mathcal{W}^k(\mathfrak{g}, f_2)$.

Proof. The proof relies on the same arguments as the proof of Theorem 6.1.2.4. $\square$

6.2.4. Proof of reduction by stages for W-algebras

Recall the BRST cochain complex $(\mathscr{C}_0, \mathscr{d}_0)$ introduced in Theorem 6.2.1.3. Geometrically, this complex corresponds to the moment map

$$\mu_0 : \widetilde{\pi}_1{}^{-1}(\widetilde{\mathbb{O}}_1)/\!/\widetilde{N}_1 \longrightarrow \mathfrak{n}_0{}^*, \quad [\xi] \longmapsto \widetilde{\pi}_2(\xi) + \overline{\chi}_0,$$

where the acting group is N_0, that is defined by Theorem 2.3.3.3. Denote the Poisson vertex BRST complex associated with the arc space of this moment map by $(C^\bullet_{0,\infty}, d_{0,\infty})$.

Proposition 6.2.4.1. *The following cohomology vanishes outside degree* 0*:*

$$\mathrm{H}^n(\mathscr{C}_0, \mathscr{d}_0) \quad \textit{for} \quad n \neq 0,$$

and there is a natural isomorphism

$$\mathrm{gr}^{\mathrm{F}}\,\mathrm{H}^0(\mathscr{C}_0, \mathscr{d}_0) \xrightarrow{\sim} \mathrm{H}^0(C^\bullet_{0,\infty}, d_{0,\infty}),$$

where the filtration F *on the cohomology* $\mathrm{H}^0(\mathscr{C}_0, \mathscr{d}_0)$ *is induced by the Li filtration on the complex* $(\mathscr{C}_0, \mathscr{d}_0)$*.*

Proof. We equip $\mathscr{W}^k(\mathfrak{g}, f_1)$ with the Hamitlonian operator $\mathscr{H}_0^{\mathrm{old}}$ induced by the operator defined in equation (6.1.3.3). Because of Theorem 6.1.3.4, we can provide a strong basis $\{J^{\{i\}}\}_{i=0}^{\ell}$ of the W-algebra corresponding to an H_1-homogeneous basis $\{x_i\}_{i=0}^{\ell}$ of $\mathfrak{g}^{f_1}$. Each element J^i is $\mathscr{H}_0^{\mathrm{old}}$-homogeneous of degree $\Delta(J^{\{i\}}) = 1 - \frac{1}{2}\delta(x_i)$, where $\delta(x_i)$ is the degree of x_i. Because $\delta(x_i)$ is nonpositive, $\Delta(J^{\{i\}})$ is positive. This implies that each homogeneous component of the W-algebra $\mathscr{W}^k(\mathfrak{g}, f_1)$ is finite dimensional.

For x in $\mathfrak{n}_0$, the image $\Upsilon_0(x)$ is the sum of terms of degrees less than one. The standard comoment map is given by $\Upsilon_{0,\mathrm{st}}(x) = J^{\{x\}}$. The induced action of $\mathcal{U}(\mathfrak{n}_0[t^{-1}]t^{-1})$ is free because the affine W-algebra is freely generated. Hence, Theorem 5.3.4.2 holds and the cohomology of the complex $(\mathscr{C}_0, \mathscr{d}_0)$ is isomorphic to the cohomology of the corresponding quotient complex $(\mathscr{C}_{0,+}, \mathscr{d}_{0,+})$.

Let $\{x_i\}_{i \in I(\mathfrak{n}_0)}$ be a basis of $\mathfrak{n}_0$ which is homogeneous for the H_2-grading and denote by $\delta(x_i)$ the degree of x_i. Let $H_0 \coloneqq H_2 - H_1$ which belongs to the Lie algebra $\mathfrak{g}^{\natural,1}$. Define the following Hamiltonian operator on the complex $\mathscr{C}_0$:

$$\mathscr{H}_0^{\mathrm{new}} = \mathscr{H}_0^{\mathrm{old}} + \frac{1}{2}\big(\partial J^{\{H_0\}}\big)_{(1)} + L^{\mathscr{G}}(m_\bullet), \quad \text{where} \quad m_\bullet = \frac{1}{2}\delta(x_\bullet).$$

The degrees of the strong generators are

$$\begin{aligned} &\Delta(J^{\{x\}}) = 1 - \frac{\delta}{2} \quad \text{for} \quad x \in \mathfrak{n}_0 \cap \mathfrak{g}^{(2)}_\delta \quad \text{and} \quad \delta \in \mathbf{Z}, \\ &\Delta(\phi_i) = 1 - \frac{\delta(x_i)}{2} \quad \text{and} \quad \Delta(\phi_i^*) = \frac{\delta(x_i)}{2}, \quad \text{for} \quad i \in I(\widetilde{\mathfrak{n}}_0). \end{aligned}$$

The element Q_0 is homogeneous of degree 1 and the quotient complex $\mathcal{C}_{0,+}$ has an induced grading which is nonnegative. Moreover, the moment map is smooth and we have the isomorphism

$$N_0 \times S_2 \cong {\mu_0}^{-1}(0)$$

given by Theorem 2.3.3.3. So we can apply Theorem 5.3.5.1 and the proof is done. $\square$

Proposition 6.2.4.2 ([Ara15, Theorem 4.17])**.** *The filtration induced on the W-algebra $\mathscr{W}^k(\mathfrak{g}, f_1)$ by the Li filtration on the BRST cochain complex $\widetilde{\mathcal{C}}_1$ coincides with its Li filtration: for all p in $\mathbf{Z}$, $\mathrm{F}^p_{\mathrm{Li}}\,\mathscr{W}^k(\mathfrak{g}, f_1) = \mathrm{H}^0\big(\mathrm{F}^p_{\mathrm{Li}}\,\widetilde{\mathcal{C}}^{\bullet}_1, d_1\big)$.*

We can now prove our main theorem, that is to the say reduction by stages.

Proof of Theorem 6.2.1.3. Recall the map $\Theta : \mathrm{H}^0(\mathcal{C}^{\bullet}_0, d_0) \to \mathrm{H}^0\big(\widetilde{\mathcal{C}}^{\bullet}_2, \widetilde{d}_2\big)$ which is defined in (6.2.1.7). This map respects the Hamiltonian gradings defined on both sides, which are nonnegative. We consider the filtrations on the cohomology induced by the Li filtrations on the cochain complexes. These filtrations are finite on each homogeneous part of the gradings, so the map Θ will be an isomorphism if the associated graded map $\operatorname{gr}\Theta$ is.

On the left-hand side, using Propositions 6.2.4.1 and 6.2.4.2, there is the isomorphism

$$\operatorname{gr}^{\mathrm{F}} \mathrm{H}^0(\mathcal{C}^{\bullet}_0, d_0) \cong \mathbf{C}[\mathrm{J}_\infty({\pi_0}^{-1}(-\overline{\chi}_0)/\!/N_0)].$$

On the right-hand side, according to Proposition 6.2.3.1, there is the isomorphism

$$\operatorname{gr}^{\mathrm{F}} \mathrm{H}^0\big(\widetilde{\mathcal{C}}^{\bullet}_2, \widetilde{d}_2\big) \cong \mathbf{C}\big[\mathrm{J}_\infty\big({\widetilde{\pi}_2}^{-1}\big(\widetilde{\mathbb{O}}_2^-\big)/\!/\widetilde{N}_2\big)\big].$$

Hence, $\operatorname{gr}\Theta$ coincides with the Poisson vertex isomorphism induced by the reduction by stages for Slodowy slices: ${\widetilde{\pi}_2}^{-1}\big(\widetilde{\mathbb{O}}_2^-\big)/\!/\widetilde{N}_2 \cong {\mu_0}^{-1}(0)/\!/N_0$. $\square$

6.3. Future work and applications

To conclude this report, we describe several open problems and research direction related to our work on reduction by stages.

6.3.1. Reduction by stages in the Kazhdan–Lusztig category

Let $\mathcal{V}$ be a vertex algebra, meaning a purely even vertex superalgebra. A *$\mathcal{V}$-module* is defined as a vector space M equipped with a $\mathbf{C}$-linear map

$$\mathcal{V} \otimes_{\mathbf{C}} M \longrightarrow M(\!(z)\!), \quad a \otimes v \longmapsto a^M(z)v := \sum_{n\in\mathbf{Z}} a^M_{(n)} v\, z^{-n-1}$$

satisfying the following axioms:

1. the equality $\mathbf{1}^M(z) = \mathrm{id}_M$ holds in $\operatorname{End}(M)[\![z, z^{-1}]\!]$,

2. for all a, b in $\mathscr{V}$ and m, n in $\mathbf{Z}$, one has

$$[a^M_{(m)}, b^M_{(n)}] = \sum_{i=0}^{\infty} \binom{m}{i} (a_{(i)}b)^M_{(m+n-i)},$$

$$(a_{(m)}b)^M_{(n)} = \sum_{i=0}^{\infty} (-1)^i \big(a^M_{(m-i)} \circ b^M_{(n+i)} - (-1)^m b^M_{(m+n-i)} \circ a^M_{(i)}\big)$$

in $\operatorname{End}(M)$.

Denote by $\mathscr{V}$-Mod the category of $\mathscr{V}$-modules.

If $\mathscr{V}$ is graded by a Hamiltonian operator $\mathscr{H}$, one can defined a *graded $\mathscr{V}$-module* as a usual $\mathscr{V}$-module equipped with a semisimple operator $\mathscr{H}_M$ such that the equation

$$[\mathscr{H}_M, a^M(z)] = z\partial_z a^M(z) + (\mathscr{H}a)^M(z)$$

holds for any a in $\mathscr{V}$. If v in M is an eigenvector; denotes by $\Delta(v)$ the corresponding eigenvalue in $\mathbf{C}$. If a is an eigenvector in $\mathscr{V}$, then for any n in $\mathbf{Z}$, one has

$$\Delta(a^M_{(n)}v) = \Delta(a) + \Delta(v) - n - 1.$$

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra. Recall that the universal affine vertex algebra $\mathscr{V}^k(\mathfrak{g})$ is graded. Denote by $\mathsf{KL}^k(\mathfrak{g})$ the *Kazhdan–Lusztig category* that was introduced in [KL93]. It is defined as the full subcategory of graded $\mathscr{V}^k(\mathfrak{g})$-modules M such that

1. the module decomposes as

$$M = \sum_{i=1}^{r} \sum_{\delta \in \mathbf{Z}_{\geqslant 0}} M(\alpha_i + \delta),$$

where $\alpha_1, \dots, \alpha_r$ are some complex numbers and $M(\alpha_i + \delta)$ denotes the $\mathscr{H}_M$-eigenspace corresponding to the eigenvalue $\alpha_i + \delta$,

2. each subspace $M(\alpha_i + \delta)$ is finite-dimensional.

Let f be a nilpotent element of $\mathfrak{g}$ and $\mathfrak{g} = \bigoplus_{\delta \in \mathbf{Z}} \mathfrak{g}_\delta$ be a good grading for f defined by the adjoint action of a semisimple element H in $\mathfrak{h}$. Let $\mathfrak{l}$ be an isotropic subspace of the symplectic space $\mathfrak{g}_1$ and set $\mathfrak{n}_\mathfrak{l} \coloneqq \mathfrak{l}^{\perp,\omega} \oplus \mathfrak{g}_{\geqslant 2}$.

According to [KRW03], quantum Hamiltonian reduction can be performed for modules. For any $\mathscr{V}^k(\mathfrak{g})$-module M, one can define a BRST complex

$$\mathscr{C}_\mathfrak{l}(M)^\bullet \coloneqq M \otimes_\mathbf{C} \mathscr{A}(\mathfrak{l}^{\perp,\omega}/\mathfrak{l}) \otimes_\mathbf{C} \mathscr{F}^\bullet(\mathfrak{n}_\mathfrak{l} \oplus \mathfrak{n}_\mathfrak{l}{}^*),$$

that is a module over $\mathscr{C}_\mathfrak{l} = \mathscr{V}^k(\mathfrak{g}) \otimes_\mathbf{C} \mathscr{A}(\mathfrak{l}^{\perp,\omega}/\mathfrak{l}) \otimes_\mathbf{C} \mathscr{F}^\bullet(\mathfrak{n}_\mathfrak{l} \oplus \mathfrak{n}_\mathfrak{l}{}^*)$, with the differential defined by $\mathscr{d}^M_\mathfrak{l} \coloneqq (Q_\mathfrak{l})^{\mathscr{C}_0(M)}_{(0)}$. We recall the following vanishing result. Denote

$$\mathrm{H}^0_f(M) \coloneqq \mathrm{H}^0\big(\mathscr{C}_\mathfrak{l}(M)^\bullet, \mathscr{d}^M_\mathfrak{l}\big),$$

it is a $\mathscr{W}^k(\mathfrak{g}, f)$-module. If M is a vertex algebra over $\mathscr{V}^k(\mathfrak{g})$, then $\mathrm{H}^0_f(M)$ is a vertex algebra over $\mathscr{W}^k(\mathfrak{g}, f)$.

Proposition 6.3.1.1 ([Ara15, Theorem 4.15]). *If M belongs to $\mathsf{KL}^k(\mathfrak{g})$, then for $n \neq 0$, one has the vanishing*

$$\mathrm{H}^n\big(\mathscr{C}_{\mathfrak{l}}(M)^\bullet, d_{\mathfrak{l}}^M\big) = 0.$$

Hence, the functor

$$\mathrm{H}^0_f : \mathsf{KL}^k(\mathfrak{g}) \longrightarrow \mathscr{W}^k(\mathfrak{g}, f)\text{-}\mathsf{Mod}$$

is exact.

For $i = 1, 2$, let f_i be a nilpotent element and H_i be in the Cartan subalgebra $\mathfrak{h}$ such that the Lie algebra grading $\mathfrak{g} = \bigoplus_{\delta \in \mathbf{Z}} \mathfrak{g}_\delta^{(i)}$ defined by the adjoint action of H_i is a good grading for f_i. If Conditions ($\bigstar$) hold, then for any $\mathscr{W}^k(\mathfrak{g}, f_1)$-module M, one can define a BRST cochain complex

$$\mathscr{C}_0(M)^\bullet := M \otimes_{\mathbf{C}} \mathscr{F}^\bullet(\mathfrak{n}_0 \oplus \mathfrak{n}_0{}^*),$$

that is a module over $\mathscr{C}_0 = \mathscr{W}^k(\mathfrak{g}, f_1) \otimes_{\mathbf{C}} \mathscr{F}^\bullet(\mathfrak{n}_0 \oplus \mathfrak{n}_0{}^*)$, with the differential defined by $d_0^M := (Q_0)_{(0)}^{\mathscr{C}_0(M)}$. Denote

$$\mathrm{H}^0_{f_0}(M) := \mathrm{H}^0\big(\mathscr{C}_0(M)^\bullet, d_0^M\big),$$

it is a $\mathscr{W}^k(\mathfrak{g}, f_2)$-module by Theorem 6.2.1.3.

Conjecture 6.3.1.2. *Let M be in $\mathsf{KL}^k(\mathfrak{g})$. If Conditions ($\bigstar$) hold, there is an isomorphism of $\mathscr{W}^k(\mathfrak{g}, f_2)$-modules:*

$$\mathrm{H}^0_{f_0} \circ \mathrm{H}^0_{f_1}(M) \cong \mathrm{H}^0_{f_2}(M).$$

Moreover, if M has a structure of vertex algebra over $\mathscr{V}^k(\mathfrak{g})$, this is an isomorphism of vertex algebra over $\mathscr{W}^k(\mathfrak{g}, f_2)$.

Reduction by stages in the Kazhdan–Lusztig category has already been established in several cases when k is a *generic level*, that is to say k is not rational: see [FFFN24, FKN24]. In this case, the category $\mathsf{KL}^k(\mathfrak{g})$ is semisimple and one can use a screening operator description of the simple modules to establish reduction by stages. Because we have a functorial reduction by stages in the geometric setting (Corollary 2.3.4.1), we want to prove Conjecture 6.3.1.2 by using geometric methods as we did to establish Theorem 4.

Arakawa already used this kind of techniques with success to establish Proposition 6.3.1.1. We want to adapt his arguments to the reduction by stages setting. To adapt the proof of Theorem 4 for any module M in $\mathsf{KL}^k(\mathfrak{g})$, two ingredients are still missing:

1. a natural map $\mathrm{H}^0_{f_0} \circ \mathrm{H}^0_{f_1}(M) \to \mathrm{H}^0_{f_2}(M)$,
2. the freeness of the action of $\mathscr{U}(\mathfrak{n}_0[t^{-1}]t^{-1})$ on $\mathrm{H}^0_{f_1}(M)$.

6.3.2. Structure of affine W-algebras in type A

For the last years, the development of reduction by stages for affine W-algebras lead several researchers (Creutzig, Fasquel, Genra, the author of this thesis, Linshaw and Nakatsuka) to conjecture that any affine W-algebra in type A can be reconstructed by using only hook-type reduction. More precisely, let $\mathfrak{g}$ be the simple Lie algebran $\mathfrak{sl}_n$ and let f be a nilpotent element corresponding to a partition $(a_1, \dots, a_r)$ of n. Denote by f_i a nilpotent element in $\mathfrak{sl}_{n_i}$ associated with the hook-type partition $(a_i, 1^{n_i - a_i})$ of $n_i \coloneqq n - a_1 - \cdots a_{i-1}$.

Conjecture 6.3.2.1 ([CFLN24, Conjecture A]). *There is a vertex algebra isomorphism*

$$\mathrm{H}^0_{f_r} \circ \cdots \circ \mathrm{H}^0_{f_1}(\mathcal{V}^k(\mathfrak{sl}_n)) \cong \mathcal{W}^k(\mathfrak{sl}_n, f).$$

For any module M in $\mathsf{KL}^k(\mathfrak{sl}_n)$, there is an isomorphism of $\mathcal{W}^k(\mathfrak{sl}_n, f)$-modules

$$\mathrm{H}^0_{f_r} \circ \cdots \circ \mathrm{H}^0_{f_1}(M) \cong \mathrm{H}^0_f(M).$$

Let us prove that the geometric analogue of this conjecture holds. We start with the following lemma.

Lemma 6.3.2.2. *Let $a_1, \dots, a_s, a, b$ be integer such that $(a_1, \dots, a_s, a, 1^b)$ is a partition of n and let f_2 be a nilpotent element of $\mathfrak{sl}_n$ corresponding to this partition. Consider the smaller partition $(a_1, \dots, a_s, 1^{a+b})$ and let f_1 be a nilpotent element corresponding to this partition. Set $p = a + b$ and denote by f_0 a nilpotent element of $\mathfrak{sl}_p$ corresponding to the hook-type partition $(a, 1^b)$.*

Then there is a unipotent subgroup N_0 of SL_p such that:

1. *the Slodowy slice S_{f_0} is the Hamiltonian reduction of $(\mathfrak{sl}_p)^*$ by N_0,*
2. *the group N_0 acts on the Slodowy slice S_{f_1} with some moment map and the corresponding Hamiltonian action is isomorphic to S_{f_2}.*

The analogous statement holds for a SL_n-variety X equipped with a moment map $X \to (\mathfrak{sl}_n)^*$.

Proof. Start with the case when $s = 0$. Then f_1, f_2 correspond respectively to the partitions $(1^n), (a, 1^b)$ with $a + b = n$, so it is simply the usual hook-type Hamiltonian reduction.

For $s > 0$, set $p = a + b$. Take the Young diagram corresponding to the partition $(a_1, \dots, a_s, 1^p)$, choose a bijective labelling of the boxes by the set $\{1, 2, \dots, n\}$.

Let f_1 be the nilpotent element constructed as the sum of elementary matrices $E_{i,j}$ such that the box labelled by i is on the same line as the box labelled by j, and the right-hand side of box i is the right-hand side of box j.

Denote by A the subset of $\{1, 2, \ldots, n\}$ that labels the boxes of the Young diagram corresponding to the subpartition (1^p). It is clear that

$$\mathfrak{gl}_p \coloneqq \operatorname{Span}\{E_{i,j} \mid i, j \in A\}$$

lies in degree 0 for the good grading corresponding to any pyramid built from this labelled Young diagram and $[\mathfrak{gl}_p, f] = 0$.

Denote by f_0 the hook-type nilpotent element of $\mathfrak{sl}_p$ corresponding to the partition $(a, 1^b)$. Consider the left-aligned pyramid corresponding to $(a, 1^b)$ and use the labelling by A introduced above. Denote by N_0 the unipotent subgroup of SL_p such that its Lie algebra $\mathfrak{n}_0$ is spanned by the $E_{i,j}$ where i, j are in A such that the box i is strictly more to the right than the box j. Then the Slodowy slice S_{f_0} in $\mathfrak{sl}_p$ is the Hamiltonian reduction of $(\mathfrak{sl}_p)^*$ by N_0.

By Proposition 2.2.6.1, we have an action of SL_p on S_{f_1} (by conjugation) and a moment map

$$\phi : S_{f_1} \longrightarrow (\mathfrak{sl}_p)^*.$$

So we get an action of N_0 on S_{f_1} and an induced moment map as explained in Section 2.2.4. We claim that the corresponding Hamiltonian reduction of S_{f_1} is isomorphic to S_{f_2}.

Note that any hook-type Hamiltonian reduction from $(\mathfrak{sl}_p)^*$ to S_{f_0}, where f_0 is of type $(a, 1^b)$, can be decomposed in a sequence of reduction by stages between "adjacent hook", meaning reduction between Slodowy slices corresponding to hook-type partitions of the form $(i, 1^j)$ and $(i + 1, 1^{j-1})$ for $1 \leqslant i \leqslant a$.

But reduction by stages is functorial (Corollary 2.3.4.1), so the reduction of S_{f_1} by N_0 also decomposes as a sequence of reduction by stages corresponding to partitions of the form $(a_1, \ldots, a_s, i, 1^j)$ and $(a_1, \ldots, a_s, i{+}1, 1^{j-1})$, $1 \leqslant i \leqslant a$. By Proposition 3.1.2.2, these reductions applied successively compute a Poisson variety that is isomorphic to S_{f_2}. $\square$

Theorem 6.3.2.3. *Let f be a nilpotent element corresponding to a partition $(a_1, \ldots, a_r)$ of n. Denote by f_i a nilpotent element in $\mathfrak{sl}_{n_i}$ associated with the hook-type partition $(a_i, 1^{n_i - a_i})$ of $n_i \coloneqq n - a_1 - \cdots a_{i-1}$, for $1 \leqslant i \leqslant r$. Then there is a sequence of unipotent groups N_i, contained in SL_{n_i}, such that*

1. *the Slodowy slice S_{f_i} is the Hamiltonian reduction of $(\mathfrak{sl}_{n_i})^*$ by N_i,*
2. *the Slodowy slice S_f is isomorphic, as a Poisson variety, to the variety resulting by computing iteratively the Hamiltonian reductions of $(\mathfrak{sl}_n)^*$ first by N_1, then by N_2, ..., and finally by N_r.*

The analogous statement holds for a SL_n-variety X equipped with a moment map $X \to (\mathfrak{sl}_n)^*$.

Proof. The theorem follows from applying inductively Lemma 6.3.2.2. $\square$

This proof can be adapted to the setting of affine W-algebras up to proving a functorial version of reduction by stages in the Kazhdan–Lusztig category. Hence Conjecture 6.3.1.2 implies Conjecture 6.3.2.1.

6.3.3. The Kraft–Procesi rule for affine W-algebras

This conjecture relates to a generalisation of the Kac–Roan–Wakimoto embedding. Let $(a_1, \ldots, a_s, a_{s+1}, \ldots, a_r)$ be a partition of n and let f_2 be a nilpotent element of $\mathfrak{sl}_n$ corresponding to this partition. Set $p \coloneqq a_{s+1} + \cdots + a_r$ and let f_1 be a nilpotent element corresponding to the partition $(a_1, \ldots, a_s, 1^p)$ of n. In general, f_1, f_2 do not satisfy Conditions ($\bigstar$). By [KRW03] (see Proposition 6.1.4.1), there is a level ℓ in $\mathbf{C}$ such that there is a vertex algebra embedding $\mathscr{V}^\ell(\mathfrak{sl}_p) \hookrightarrow \mathscr{W}^k(\mathfrak{sl}_n, f_1)$. Let f_0 be a nilpotent element of $\mathfrak{sl}_p$ corresponding to the partition $(a_{s+1}, \ldots, a_r)$ of p.

Conjecture 6.3.3.1. *There is a vertex algebra embedding*

$$\mathscr{W}^\ell(\mathfrak{sl}_p, f_0) \longhookrightarrow \mathscr{W}^k(\mathfrak{sl}_n, f_2).$$

The analogue of Conjecture 6.3.3.1 for finite W-algebras is known and plays an important role for their representation theory, see [FRZ20]. It follows from the description of finite W-algebras as truncated shifted Yangians [BK06]. Conjecture 6.3.3.1 can be deduced from Conjecture 6.3.2.1 by applying iteratively hook-type Hamiltonian reductions to the Kac–Roan–Wakimoto embedding $\mathscr{V}^\ell(\mathfrak{sl}_p) \hookrightarrow \mathscr{W}^k(\mathfrak{sl}_n, f_1)$.

A particular case of this conjecture was established by Allegra in his thesis. For the proof, he used an explicit description of the involved W-algebras by generators and relations.

Lemma 6.3.3.2 ([All19, Lemma 4.1])**.** *Let f_0 be a nilpotent element in $\mathfrak{sl}_4$ corresponding to the partition $(2,2)$ of 4 and f_2 in $\mathfrak{sl}_7$ corresponding to the partition $(3,2,2)$ of 7. There is a vertex algebra embedding*

$$\mathscr{W}^{-4+4/3}(\mathfrak{sl}_4, f_0) \longhookrightarrow \mathscr{W}^{-7+7/3}(\mathfrak{sl}_7, f_2).$$

The motivation of Allegra for this result was the following conjecture of Arakawa, van Ekeren and Moreau. Assume that the complex number k is *admissible*, of the form:

$$k = -n + \frac{n}{q}, \quad q \in \mathbf{Z}_{\geqslant 1}, \quad q, n \text{ coprime}.$$

Denote by $\mathscr{L}_k(\mathfrak{g})$ the (unique) graded simple quotient of $\mathscr{V}^k(\mathfrak{g})$ and by $\mathscr{W}_k(\mathfrak{g}, f)$ the (unique) graded simple quotient of $\mathscr{W}^k(\mathfrak{g}, f)$.

Let f_2 be a nilpotent element corresponding to a partition of n of the shape $(q^m, a_\bullet^{(0)})$, where $a_\bullet^{(0)}$ is a partition of $p \coloneqq n - qm$, and f_0 be a nilpotent element in $\mathfrak{sl}_p$ corresponding to the partition $a_\bullet^{(0)}$.

Conjecture 6.3.3.3 ([AvEM24, Conjecture 8.11])**.** *There is a vertex algebra isomorphism:*

$$\mathscr{W}_{-p+p/q}(\mathfrak{sl}_p, f_0) \cong \mathscr{W}_{-n+n/q}(\mathfrak{sl}_n, f_2).$$

Allegra proved a particular case of the conjecture as a consequence of Lemma 6.3.3.2. It is also proved in [AvEM24] by other techniques, see the discussion after their Conjecture 8.11.

Proposition 6.3.3.4 ([All19, Theorem 4.5])**.** *Let f_0 be a nilpotent element in $\mathfrak{sl}_4$ corresponding to the partition $(2,2)$ of 4 and f_2 in $\mathfrak{sl}_7$ corresponding to the partition $(3,2,2)$ of 7. There is a vertex algebra isomorphism*

$$\mathscr{W}_{-4+4/3}(\mathfrak{sl}_4, f_0) \cong \mathscr{W}_{-7+7/3}(\mathfrak{sl}_7, f_2).$$

Arakawa, van Ekeren and Moreau proved the following cases of the conjecture. Let f_1 be a nilpotent element corresponding to a partition of the form $(q^m, 1^p)$.

Proposition 6.3.3.5 ([AvEM24, Theorem 1.2])**.** *There is a vertex algebra isomorphism:*

$$\mathscr{L}_{-p+p/q}(\mathfrak{sl}_p) \cong \mathscr{W}_{-n+n/q}(\mathfrak{sl}_n, f_1).$$

We expect to use reduction by stages to deduce Conjecture 6.3.3.3 from Proposition 6.3.3.5. Let us explain the strategy.

Recall that k is called a *collapsing level* for the simple Lie algebra $\mathfrak{g}$ and the nilpotent element f if the Kac–Roan–Wakimoto embedding (Proposition 6.1.4.1)

$$\mathscr{V}^{\tau_k}(\mathfrak{g}^\natural) \longrightarrow \mathscr{W}^k(\mathfrak{g}, f)$$

induces an isomorphism after taking quotients:

$$\mathscr{L}_{\tau_k}(\mathfrak{g}^\natural) \cong \mathscr{W}_k(\mathfrak{g}, f). \tag{6.3.3.6}$$

Proposition 6.3.3.5 give examples of admissible levels that are collapsing.

Now denote by $\mathfrak{g}^\natural_{\mathrm{red}}$ the maximal reductive subalgebra of $\mathfrak{g}^\natural$. Let f_0 be a nilpotent element in $\mathfrak{g}^\natural_{\mathrm{red}}$ such that $f_2 := f + f_0$ is nilpotent in $\mathfrak{g}$ and such that there is a reduction by stages isomorphism

$$\mathscr{W}^k(\mathfrak{g}, f_2) \cong \mathrm{H}^0_{f_0}(\mathscr{W}^k(\mathfrak{g}, f)).$$

By applying the functor $\mathrm{H}^0_{f_0}$ to the isomorphism (6.3.3.6), we deduce a vertex algebra isomorphism

$$\mathscr{W}_{\tau_k}(\mathfrak{g}^\natural, f_0) \cong \mathscr{W}_k(\mathfrak{g}, f_2),$$

where $\mathscr{W}_{\tau_k}(\mathfrak{g}^\natural, f_0)$ is the graded simple quotient of $\mathscr{W}^{\tau_k}(\mathfrak{g}^\natural, f_0) := \mathrm{H}^0_{f_0}(\mathscr{V}^{\tau_k}(\mathfrak{g}^\natural))$.

Remark 6.3.3.7. There is some subtlety in this strategy. It is conjectured, but not proved in general, that the simple quotient of quantum Hamiltonian reduction is the quantum Hamiltonian reduction of the quotient:

$$\mathscr{W}_k(\mathfrak{g}, f) \cong \mathrm{H}^0_f(\mathscr{L}_k(\mathfrak{g})) \quad \text{and} \quad \mathscr{W}_{\tau_k}(\mathfrak{g}^\natural, f_0) \cong \mathrm{H}^0_{f_0}(\mathscr{L}_{\tau_k}(\mathfrak{g}^\natural)),$$

where $\mathscr{L}_{\tau_k}(\mathfrak{g}^\natural)$ is the graded simple quotient of $\mathscr{V}^{\tau_k}(\mathfrak{g}^\natural)$. However, in the setting of Conjecture 6.3.3.3 and Proposition 6.3.3.5, this is true [AvE23, Theorem 7.8].

Conjecture 6.3.3.3 and Proposition 6.3.3.5 are motivated by the following geometric results. As mentioned in Section 1.2.3, to any vertex algebra $\mathscr{V}$ one can associate a Poisson variety $X_{\mathscr{V}}$, called the *associated variety* of $\mathscr{V}$ [Ara12]. In [Ara15], the following associated varieties are computed

Denote by $\overline{\mathbf{O}_q}$ denotes the closure of the nilpotent coadjoint orbit corresponding to the partition $(q^d, 1^e)$, where $n = dq + e$ is the euclidean division of n by q. The associated variety of $\mathscr{W}_{-n+n/q}(\mathfrak{sl}_n, f_1)$ and $\mathscr{W}_{-n+n/q}(\mathfrak{sl}_n, f_2)$ are given by the nilpotent Slodowy slices $\overline{\mathbf{O}_q} \cap S_{f_1}$ and $\overline{\mathbf{O}_q} \cap S_{f_2}$ respectively. The associated variety of $\mathscr{L}_{-p+p/q}(\mathfrak{sl}_p)$ is given by $\overline{\mathbf{O}'_q}$, the closure of the nilpotent orbit corresponding to the partition $(q^{d'}, 1^{e'})$, where $s = d'q + e'$ is the euclidean division of s by q. The associated variety of $\mathscr{W}_{-p+p/q}(\mathfrak{sl}_p, f_0)$ is the nilpotent Slodowy slice $\overline{\mathbf{O}'_q} \cap S_{f_0}$.

In their work to classify singularities of nilpotent orbit closures in type A, Kraft and Procesi established *smooth equivalences* between some nilpotent Slodowy slices [KP81]. By [KP81, Proposition 3.1], $\overline{\mathbf{O}_q} \cap S_{f_1}$ is smoothly equivalent to $\overline{\mathbf{O}'_q}$, and $\overline{\mathbf{O}_q} \cap S_{f_2}$ is smoothly equivalent to $\overline{\mathbf{O}'_q} \cap S_{f_0}$. Actually, these nilpotent Slodowy slices are isomorphic quiver varieties [Maf05, Hen15]. Note that the isomorphism $\overline{\mathbf{O}_q} \cap S_{f_1} \cong \overline{\mathbf{O}'_q}$ implies $\overline{\mathbf{O}_q} \cap S_{f_2} \cong \overline{\mathbf{O}'_q} \cap S_{f_0}$ by applying Theorem 6.3.2.3. This gives an evidence to support our strategy to prove Conjecture 6.3.3.3.

In other classical types, Kraft and Procesi have also provided equivalences of singularities between nilpotent Slodowy slices [KP82]. In exceptional types, some smooth equivalences have been proved by Fu, Juteau, Levy and Sommers [FJLS17]. According to these works, analogues of Conjecture 6.3.3.3 and Proposition 6.3.3.5 have been stated in [AvEM24] for the other types. We expect that reduction by stages could also be used in other types, since we have examples. But it seems that Conditions ($\bigstar$) do not provide enough examples in other types for this purpose, so they must be relaxed. It is expected that reduction by stages happens for any pair of nilpotent orbits $\mathbf{O}_1, \mathbf{O}_2$ in a simple Lie algebra $\mathfrak{g}$ satisfying the inclusion $\overline{\mathbf{O}_1} \subseteq \overline{\mathbf{O}_2}$ and we hope to prove such a result in the future.

Bibliography

[ACET24] Filippo Ambrosio, Giovanna Carnovale, Francesco Esposito, and Lewis Topley. Universal filtered quantizations of nilpotent Slodowy slices. *Journal of noncommutative geometry*, 18(1):1–35, 2024.

[ACG24] Dražen Adamović, Thomas Creutzig, and Naoki Genra. Relaxed and logarithmic modules of $\widehat{\mathfrak{sl}_3}$. *Mathematische Annalen*, 389(1):281–324, 2024.

[Ada19] Dražen Adamović. Realizations of simple affine vertex algebras and their modules: The cases $\widehat{\mathfrak{sl}(2)}$ and $\widehat{\mathfrak{osp}(2)}$. *Communications in Mathematical Physics*, 366(3):1025–1067, 2019.

[AF19] Tomoyuki Arakawa and Edward Frenkel. Quantum langlands duality of representations of $\mathscr{W}$-algebras. *Compositio Mathematica*, 155(12):2235–2262, 2019.

[AFP22] Dražen Adamović, Pierluigi Möseneder Frajria, and Paolo Papi. On the semisimplicity of the category KL_k for affine Lie superalgebras. *Advances in mathematics*, 405:108493, 2022.

[AKM15] Tomoyuki Arakawa, Toshiro Kuwabara, and Fyodor Malikov. Localization of affine W-algebras. *Communications in Mathematical Physics*, 335(1):143–182, 2015.

[AKMF$^+$20] Dražen Adamović, Victor G. Kac, Pierluigi Möseneder Frajria, Paolo Papi, and Ozren Perše. An application of collapsing levels to the representation theory of affine vertex algebras. *International mathematics research notices*, 2020(13):4103–4143, 2020.

[AKR21] Dražen Adamović, Kazuya Kawasetsu, and David Ridout. A realisation of the Bershadsky–Polyakov algebras and their relaxed modules. *Letters in mathematical physics*, 111(2):38, 2021.

[All19] Francesco Allegra. *W-algebras in type A and the Arakawa–Moreau conjecture*. PhD thesis, Università di Roma (Sapienza), 2019.

[AM18] Tomoyuki Arakawa and Anne Moreau. Joseph ideals and lisse minimal W-algebras. *Journal of the Institute of Mathematics of Jussieu*, 17(2):397–417, 2018.

[AM25] Tomoyuki Arakawa and Anne Moreau. *Arc spaces and vertex algebras.* 2025. Premilinary version, https://www.imo.universite-paris-saclay.fr/~anne.moreau.

[Ara07] Tomoyuki Arakawa. Representation theory of $\mathscr{W}$-algebras. *Invent. Math.*, 169(2):219–320, 2007.

[Ara12] Tomoyuki Arakawa. A remark on the C_2-cofiniteness condition on vertex algebras. *Math. Z.*, 270(1-2):559–575, 2012.

[Ara15] Tomoyuki Arakawa. Associated varieties of modules over Kac–Moody algebras and C_2-cofiniteness of W-algebras. *Int. Math. Res. Not. IMRN*, (22):11605–11666, 2015.

[Ara18] Tomoyuki Arakawa. Associated varieties and Higgs branches (a survey). *Contemp. Math*, 711:37–44, 2018.

[Ara19] Tomoyuki Arakawa. Chiral algebras of class $\mathscr{S}$ and Moore–Tachikawa symplectic varieties, 2019. Preprint, arXiv:1811.01577.

[AvE23] Tomoyuki Arakawa and Jethro van Ekeren. Rationality and fusion rules of exceptional W-algebras. *Journal of the European Mathematical Society (EMS Publishing)*, 25(7), 2023.

[AvEM24] Tomoyuki Arakawa, Jethro van Ekeren, and Anne Moreau. Singularities of nilpotent Slodowy slices and collapsing levels of W-algebras. In *Forum of Mathematics, Sigma*, volume 12, page e95. Cambridge University Press, 2024.

[BFM23] Tristan Bozec, Maxime Fairon, and Anne Moreau. Functorial constructions related to double Poisson vertex algebras, 2023. Preprint, arXiv:2307.06071.

[BG07] Jonathan Brundan and Simon M. Goodwin. Good grading polytopes. *Proceedings of the London Mathematical Society*, 94(1):155–180, 2007.

[BGK08] Jonathan Brundan, Simon M. Goodwin, and Alexander Kleshchev. Highest weight theory for finite W-algebras. *International Mathematics Research Notices*, 2008(9):rnn051–rnn051, 2008.

[BK06] Jonathan Brundan and Alexander Kleshchev. Shifted Yangians and finite W-algebras. *Advances in Mathematics*, 200(1):136–195, 2006.

[BLL+15] Christopher Beem, Madalena Lemos, Pedro Liendo, Wolfger Peelaers, Leonardo Rastelli, and Balt C. Van Rees. Infinite chiral symmetry in four dimensions. *Communications in Mathematical Physics*, 336:1359–1433, 2015.

[Bor86] Richard E. Borcherds. Vertex algebras, Kac–Moody algebras, and the Monster. *Proceedings of the National Academy of Sciences - PNAS*, 83(10):3068–3071, 1986.

[Bor87] Armand Borel. *Algebraic D-modules*. Perspectives in mathematics. Academic Press, 1987.

[Bor92] Richard E. Borcherds. Monstrous Moonshine and Monstrous Lie superalgebras. *Inventiones mathematicae*, 109(1):405–444, 1992.

[BPW16] Tom Braden, Nicholas Proudfoot, and Ben Webster. Quantizations of conical symplectic resolutions I: local and global structure. *Astérisque*, 384:1–73, 2016.

[BPZ84] Alexander A. Belavin, Alexander M. Polyakov, and Alexander B. Zamolodchikov. Infinite conformal symmetry in two-dimensional Quantum Field Theory. *Nuclear Physics B*, 241(2):333–380, 1984.

[BS93] Peter Bouwknegt and Kareljan Schoutens. W-symmetry in conformal field theory. *Physics Reports*, 223(4):183–276, 1993.

[But23] Dylan Butson. Vertex algebras from divisors on Calabi–Yau threefolds. 2023. Preprint, arXiv:2312.03648.

[CdS01] Ana Cannas da Silva. *Lectures on symplectic geometry*, volume 3575. Springer, 2001.

[CFLN24] Thomas Creutzig, Justine Fasquel, Andrew R. Linshaw, and Shigenori Nakatsuka. On the structure of W-algebras in type A. 2024. Preprint, arXiv:2403.08212.

[CL18] Thomas Creutzig and Andrew R. Linshaw. Cosets of the $\mathcal{W}^k(\mathfrak{sl}_4, f_{\text{subreg}})$-algebra. *Contemp. Math*, 711:105–117, 2018.

[CLNS18] Antoine Chambert-Loir, Johannes Nicaise, and Julien Sebag. *Motivic integration*. Springer, 2018.

[CM93] David H. Collingwood and William M. McGovern. *Nilpotent orbits in semisimple Lie algebras*. Van Nostrand Reinhold Co. New York, 1993.

[DDCDS+06] Alessandro D'Andrea, Corrado De Concini, Alberto De Sole, Reimundo Heluani, and Victor G. Kac. Three equivalent definitions of finite W-algebras, 2006. Appendix to [DSK06].

[DSK05] Alberto De Sole and Victor G. Kac. Freely generated vertex algebras and non–linear Lie conformal algebras. *Communications in mathematical physics*, 254:659–694, 2005.

[DSK06] Alberto De Sole and Victor G. Kac. Finite vs affine W-algebras. *Japanese Journal of Mathematics*, 1(1):137–261, 2006.

[DSKV16] Alberto De Sole, Victor Kac, and Daniele Valeri. Structure of classical (finite and affine) W-algebras. *Journal of the European Mathematical Society*, 18(9):1873–1908, 2016.

[Eis13] David Eisenbud. *Commutative algebra: with a view toward algebraic geometry*, volume 150. Springer Science & Business Media, 2013.

[EK05] Alexander G. Elashvili and Victor G. Kac. Classification of good gradings of simple Lie algebras. *Translations of the American Mathematical Society-Series 2*, 213:85–104, 2005.

[Fas22] Justine Fasquel. Rationality of the exceptional W-algebras $\mathcal{W}^k(\mathfrak{sp}_4, f_{\mathrm{subreg}})$ associated with subregular nilpotent elements of $\mathfrak{sp}_4$. *Communications in Mathematical Physics*, 390(1):33–65, 2022.

[Fas25] Justine Fasquel. OPEs of rank two W-algebras. *Contemporary Mathematics*, 813, 2025.

[FBZ04] Edward Frenkel and David Ben-Zvi. *Vertex algebras and algebraic curves*, volume 88 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2 edition, 2004.

[Feh23] Zachary Fehily. Subregular W-algebras of type A. *Communications in Contemporary Mathematics*, 25(09):2250049, 2023.

[Feh24] Zachary Fehily. Inverse reduction for hook-type W-algebras. *Communications in Mathematical Physics*, 405(9):214, 2024.

[FF90] Boris Feigin and Edward Frenkel. Quantization of the Drinfeld–Sokolov reduction. *Physics Letters B*, 246(1-2):75–81, 1990.

[FFFN24] Justine Fasquel, Zachary Fehily, Ethan Fursman, and Shigenori Nakatsuka. Connecting affine W-algebras: a case study on $\mathfrak{sl}_4$. 2024. Preprint, arXiv:2408.13785.

[FJLS17] Baohua Fu, Daniel Juteau, Paul Levy, and Eric Sommers. Generic singularities of nilpotent orbit closures. *Advances in Mathematics*, 305:1–77, 2017.

[FKN24] Justine Fasquel, Vladimir Kovalchuk, and Shigenori Nakatsuka. On Virasoro-type reductions and inverse Hamiltonian reductions for W-algebras and W_∞-algebras. 2024. Preprint, arXiv:2411.10694.

[FLM89] Igor Frenkel, James Lepowsky, and Arne Meurman. *Vertex operator algebras and the Monster*, volume 134. Academic press, 1989.

[FN23] Justine Fasquel and Shigenori Nakatsuka. Orthosymplectic Feigin–Semikhatov duality. 2023. Preprint, arXiv:2307.14574.

[Fre07] Edward Frenkel. *Langlands correspondence for loop groups*, volume 103. Cambridge University Press Cambridge, 2007.

[FRZ20] Vyacheslav Futorny, Luis Enrique Ramirez, and Jian Zhang. Gelfand–Tsetlin representations of finite W-algebras. *Journal of Pure and Applied Algebra*, 224(5):106226, 2020.

[FZ92] Igor Frenkel and Yongchang Zhu. Vertex operator algebras associated to representations of affine and Virasoro algebras. *Duke mathematical journal*, 66(1):123–168, 1992.

[Gen20] Naoki Genra. Screening operators and parabolic inductions for affine W-algebras. *Advances in Mathematics*, 369:107179, 2020.

[GG02] Wee Liang Gan and Victor Ginzburg. Quantization of Slodowy slices. *International Mathematics Research Notices*, 2002(5):243–255, 2002.

[GG06] Wee Liang Gan and Victor Ginzburg. Almost-commuting variety, D-modules, and Cherednik algebras. *International Mathematics Research Papers*, 2006, 2006.

[Gin09] Victor Ginzburg. Harish-Chandra bimodules for quantized Slodowy slices. *Representation Theory of the American Mathematical Society*, 13(12):236–271, 2009.

[GJ24] Naoki Genra and Thibault Juillard. Reduction by stages for finite W-algebras. *Mathematische Zeitschrift*, 308(1):15, 2024.

[GJ25] Naoki Genra and Thibault Juillard. Reduction by stages for affine W-algebras. 2025. Preprint, arXiv:2501.04501.

[Hen15] Anthony Henderson. Singularities of nilpotent orbit closures, 2015. Lecture notes, arXiv:1408.3888.

[Hum72] James E. Humphreys. *Introduction to Lie algebras and representation theory*, volume Vol. 9 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1972.

[Jos85] Anthony Joseph. On the associated variety of a primitive ideal. *Journal of algebra*, 93(2):509–523, 1985.

[Kac98] Victor Kac. *Vertex algebras for beginners*, volume 10 of *University Lecture Series*. American Mathematical Society, Providence, RI, 2 edition, 1998.

[Kas03] Masaki Kashiwara. *D-modules and microlocal calculus*, volume 217. American Mathematical Soc., 2003.

[KL93] David Kazhdan and George Lusztig. Tensor structures arising from affine Lie algebras. I. *Journal of the American Mathematical Society*, 6(4):905–947, 1993.

[Kos78] Bertram Kostant. On Whittaker vectors and representation theory. *Inventiones mathematicae*, 48(2):101–184, 1978.

[KP81] Hanspeter Kraft and Claudio Procesi. Minimal singularities in GL_n. *Invent. math*, 62(3):503–515, 1981.

[KP82] Hanspeter Kraft and Claudio Procesi. On the geometry of conjugacy classes in classical groups. *Commentarii Mathematici Helvetici*, 57:539–602, 1982.

[KPW22] Joel Kamnitzer, Khoa Pham, and Alex Weekes. Hamiltonian reduction for affine Grassmannian slices and truncated shifted Yangians. *Advances in Mathematics*, 399:108281, 2022.

[KRW03] Victor Kac, Shi-Shyr Roan, and Minoru Wakimoto. Quantum reduction for affine superalgebras. *Communications in Mathematical Physics*, 241(2-3):307–342, sep 2003.

[KS87] Bertram Kostant and Shlomo Sternberg. Symplectic reduction, BRS cohomology, and infinite-dimensional Clifford algebras. *Annals of Physics*, 176(1):49–113, 1987.

[KW04] Victor Kac and Minoru Wakimoto. Quantum reduction and representation theory of superconformal algebras. *Advances in Mathematics*, 185(2):400–458, 2004.

[LGPV12] Camille Laurent-Gengoux, Anne Pichereau, and Pol Vanhaecke. *Poisson structures*, volume 347. Springer Science & Business Media, 2012.

[Li05] Haisheng Li. Abelianizing vertex algebras. *Comm. Math. Phys.*, 259(2):391–411, 2005.

[Los10] Ivan Losev. Quantized symplectic actions and W-algebras. *Journal of the American Mathematical Society*, 23(1):35–59, 2010.

[LSS15] Andrew R. Linshaw, Gerald W. Schwarz, and Bailin Song. Jet schemes and invariant theory. *Annales de l'Institut Fourier*, 65(6):2571–2599, 2015.

[Lyn79] Thomas Emile Lynch. *Generalized Whittaker vectors and representation theory.* PhD thesis, Massachusetts Institute of Technology, 1979.

[Maf05] Andrea Maffei. Quiver varieties of type A. *Commentar II Mathematici Helvetici*, 80(1):1–27, 2005.

[Mas25] Elisabetta Masut. A monoidal category viewpoint for translation functors for finite W-algebras. *Journal of Algebra*, 667:1–34, 2025.

[Mil17] James S. Milne. *Algebraic groups: the theory of group schemes of finite type over a field.* Cambridge studies in advanced mathematics 170. Cambridge University Press, 2017.

[MMO+07] Jerrold E. Marsden, Gerard Misiolek, Juan-Pablo Ortega, Matthew Perlmutter, and Tudor S Ratiu. *Hamiltonian reduction by stages.* Springer, 2007.

[Mor15a] Stephen Morgan. *Quantum Hamiltonian reduction of W-algebras and category O.* PhD thesis, University of Toronto, 2015. ArXiv:1502.07025.

[Mor15b] Stephen Morgan. Quantum Hamiltonian reduction of W-algebras and category O, 2015. Preprint, arXiv:1510.07352.

[MR97] Jens Ole Madsen and Eric Ragoucy. Secondary quantum Hamiltonian reductions. *Communications in mathematical physics*, 185(3):509–541, 1997.

[MV07] Ivan Mirković and Maxim Vybornov. Quiver varieties and Beilinson–Drinfeld Grassmannians of type A. 2007. Preprint, arXiv:0712.4160.

[MW74] Jerrold Marsden and Alan Weinstein. Reduction of symplectic manifolds with symmetry. *Reports on mathematical physics*, 5(1):121–130, 1974.

[NO06] Constantin Nastasescu and Fred van Oystaeyen. *Graded and filtered rings and modules*, volume 758 of *Lecture Notes in Mathematics*. Springer Berlin / Heidelberg, Berlin, Heidelberg, 1979 edition. edition, 2006.

[Pre02] Alexander Premet. Special transverse slices and their enveloping algebras. *Advances in Mathematics*, 170(1):1–55, 2002.

[Pre07] Alexander Premet. Enveloping algebras of Slodowy slices and the Joseph ideal. *Journal of the European Mathematical Society*, 9(3):487–543, 2007.

[Sad16] Guilnard Sadaka. Paires admissibles d'une algèbre de Lie simple complexe et W-algèbres finies. In *Annales de l'Institut Fourier*, volume 66, pages 833–870, 2016.

[Skr02] Serge Skryabin. A category equivalence, 2002. Appendix of [Pre02].

[Slo06] Peter Slodowy. *Simple Singularities and Simple Algebraic Groups*, volume 815 of *Lecture Notes in Mathematics*. Springer Berlin / Heidelberg, Berlin, Heidelberg, 1980 edition. edition, 2006.

[SV13] Olivier Schiffmann and Eric Vasserot. Cherednik algebras, W-algebras and the equivariant cohomology of the moduli space of instantons on A_2. *Publications mathématiques de l'IHÉS*, 118(1):213–342, 2013.

[SXY24] Peng Shan, Dan Xie, and Wenbin Yan. Modularity for $\mathscr{W}$-algebras and affine Springer fibres, 2024. Preprint, arXiv:2404.00760.

[Vor93] Alexander A. Voronov. Semi-infinite homological algebra. *Invent. Math.*, 113(1):103–146, 1993.

[Wei97] Charles A. Weibel. *An introduction to homological algebra*. Cambridge studies in advanced mathematics 38. Cambridge University Press, Cambridge [etc], repr. edition, 1997.

[Zam85] Alexander B. Zamolodchikov. Infinite additional symmetries in two-dimensional conformal quantum field theory. *Theoretical and mathematical physics*, 65(3):1205–1213, 1985.